

ON THE GRADED QUOTIENTS OF THE RING OF FRICKE CHARACTERS OF A FREE GROUP

Dedicated to Professor Sadayoshi Kojima on the occasion of his 60th birthday

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ABSTRACT. In this paper, for a group G , we consider an $\text{Aut } G$ -invariant ideal J generated by $\text{tr } x - 2$ for any $x \in G$ in the ring of Fricke characters of G . We study a descending filtration $J \supset J^2 \supset J^3 \supset \cdots$, and its graded quotients $\text{gr}^k(J) := J^k/J^{k+1}$ for $k \geq 1$. The first purpose of the paper is to determine the structure of $\text{gr}^k(J)$ for $G = F_n$ and $k = 1, 2$.

Next, we introduce a normal subgroup $\mathcal{E}_G(k)$ consisting of automorphisms of G which act on J/J^{k+1} trivially. These normal subgroups define a central filtration of $\text{Aut } G$. This is a Fricke character analogue of the Andreadakis-Johnson filtration $\mathcal{A}_G(k)$ of $\text{Aut } G$. The main purpose of the paper is to show that $\mathcal{E}_{F_n}(1)$ is equal to $\text{Inn } F_n \cdot \mathcal{A}_{F_n}(2)$ where $\text{Inn } F_n$ is the inner automorphism group of a free group F_n , and that $\mathcal{A}_{F_n}(2k) \subset \mathcal{E}_{F_n}(k)$ for any $k \geq 1$.

1. INTRODUCTION

Let G be a group generated by elements x_1, \dots, x_n . We denote by

$$R(G) := \text{Hom}(G, \text{SL}(2, \mathbf{C}))$$

the set of all group homomorphisms from G to $\text{SL}(2, \mathbf{C})$. Let

$$\mathcal{F}(R(G), \mathbf{C}) := \{\chi : R(G) \rightarrow \mathbf{C}\}$$

be the set of all complex-valued functions of $R(G)$. Then we can consider $\mathcal{F}(R(G), \mathbf{C})$ as a commutative ring in a natural way. For any $x \in G$, we define an element $\text{tr } x \in \mathcal{F}(R(G), \mathbf{C})$ to be

$$(\text{tr } x)(\rho) := \text{tr } \rho(x)$$

for any $\rho \in R(G)$. Here “tr” in the right hand side means the trace of 2×2 matrix $\rho(x) \in \text{SL}(2, \mathbf{C})$. The element $\text{tr } x$ in $\mathcal{F}(R(G), \mathbf{C})$ is called the Fricke character of $x \in G$. Let $\mathfrak{X}(G)$ be the \mathbf{Z} -submodule of $\mathcal{F}(R(G), \mathbf{C})$ generated by all $\text{tr } x$ for $x \in G$. Then $\mathfrak{X}(G)$ is closed under the multiplication of $\mathcal{F}(R(G), \mathbf{C})$. (See Subsection 4.1 for details.)

Classically, Fricke characters were begun to studied by Fricke for a free group F_n on x_1, \dots, x_n in connection with certain problems in the theory of Riemann surfaces. (See [3].) In 1970, Horowitz [5] and [6] investigated algebraic properties of $\mathfrak{X}(G)$ using the combinatorial group theory. In particular, he [5] showed that for any $x \in G$, the Fricke character $\text{tr } x$ can be written as a polynomial with integral coefficients in $2^n - 1$

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characters $\text{tr } x_{i_1} x_{i_2} \cdots x_{i_l}$ for $1 \leq l \leq n$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. He [6] also showed that the subgroup of $\text{Aut } F_n$ consisting of automorphisms which act on $\mathfrak{X}(F_n)$ trivially is just the inner automorphism group $\text{Inn } F_n$ of F_n . Namely, the action of $\text{Aut } F_n$ on the ring of Fricke characters $\mathfrak{X}(F_n)$ induces a faithful representation of the outer automorphism group $\text{Out } F_n := \text{Aut } F_n / \text{Inn } F_n$. However, since the rank of $\mathfrak{X}(F_n)$ as a \mathbf{Z} -module is not finite in general, it is not so easy to study this representation directly.

On the other hand, in order to make the structure of the Fricke characters $\mathfrak{X}(F_n)$ clear, it is important to study the ideal of polynomials in the characters which vanish on any representations of G . More precisely, consider a polynomial ring

$$\mathbf{Z}[t] := \mathbf{Z}[t_{i_1 \dots i_l} \mid 1 \leq l \leq n, 1 \leq i_1 < i_2 < \cdots < i_l \leq n]$$

of $2^n - 1$ indeterminates, and an ideal

$$I = \{f \in \mathbf{Z}[t] \mid f(\text{tr } \rho(x_{i_1} \cdots x_{i_l})) = 0 \text{ for any } \rho \in R(G)\}.$$

In [5], for $G = F_n$, Horowitz showed that I is trivial for $n = 1$ and 2 , and is principal for $n = 3$. Whittemore [17] showed that I is not principal for $G = F_n$ and $n \geq 4$. Although the ideal I has been studied by many authors for over forty years, very little is known for it.

Here, we consider the rationalization of the situation above. Let $\mathfrak{X}_{\mathbf{Q}}(G)$ be a \mathbf{Q} -subspace of $\mathcal{F}(R(G), \mathbf{C})$ generated by $\text{tr } x$ for any $x \in G$. Similarly to $\mathfrak{X}(G)$, $\mathfrak{X}_{\mathbf{Q}}(G)$ is closed under the multiplication of $\mathcal{F}(R(G), \mathbf{C})$, and has a multiplicative unit $1 = \frac{1}{2} \text{tr } 1_G$. Hence, $\mathfrak{X}_{\mathbf{Q}}(G)$ is a ring. We call $\mathfrak{X}_{\mathbf{Q}}(G)$ the ring of Fricke characters of G over \mathbf{Q} . By observing the formula (6) and Horowitz's result, we see that for any $x \in G$, the Fricke character $\text{tr } x$ can be written as a polynomial with rational coefficients in $n + \binom{n}{2} + \binom{n}{3}$ characters $\text{tr } x_{i_1} x_{i_2} \cdots x_{i_l}$ for $1 \leq l \leq 3$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Consider a polynomial ring

$$\mathbf{Q}[t] := \mathbf{Q}[t_{i_1 \dots i_l} \mid 1 \leq l \leq 3, 1 \leq i_1 < i_2 < \cdots < i_l \leq n]$$

and its ideal

$$I_{\mathbf{Q}} := \{f \in \mathbf{Q}[t] \mid f(\text{tr } \rho(x_{i_1} \cdots x_{i_l})) = 0 \text{ for any } \rho \in R(G)\}.$$

Similarly to I , the ideal $I_{\mathbf{Q}}$ plays important roles in the various study of the ring structure of $\mathfrak{X}_{\mathbf{Q}}(G)$. One of the most advantages to consider the rationalization of the Fricke characters is that the number of the indeterminates of $\mathbf{Q}[t]$ is fewer than that of $\mathbf{Z}[t]$, and it makes various computation much easy to handle.

In the present paper, in order to construct finite dimensional representations of $\text{Aut } G$, we consider a descending filtration of $\text{Aut } G$ -invariant ideals of $\mathbf{Q}[t]/I_{\mathbf{Q}}$, and take its graded quotients. Set $t'_{i_1 \dots i_l} := t_{i_1 \dots i_l} - 2 \in \mathbf{Q}[t]$. We also denote by $t'_{i_1 \dots i_l}$ its coset class in $\mathbf{Q}[t]/I_{\mathbf{Q}}$. Consider an ideal

$$J := (t'_{i_1 \dots i_l} \mid 1 \leq l \leq 3, 1 \leq i_1 < i_2 < \cdots < i_l \leq n) \subset \mathbf{Q}[t]/I_{\mathbf{Q}}$$

generated by all $t'_{i_1 \dots i_l}$'s. Then, we have a descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots$$

of $\text{Aut } G$ -invariant ideals of $\mathbf{Q}[t]/I_{\mathbf{Q}}$. (See Subsection 4.1 for details.) Set

$$\text{gr}^k(J) := J^k/J^{k+1}.$$

Each of $\text{gr}^k(J)$ is $\text{Aut } G$ -invariant \mathbf{Q} -vector space of finite dimension for any $k \geq 1$. This technique is deeply inspired by a result of Magnus [12] who originally studied the behavior of the action of $\text{Aut } F_3$ on $\text{gr}^1(J)$. In [12], he pointed out the difficulties to find $\text{Aut } F_n$ -invariant ideals of $\mathfrak{X}(F_n)$ and its quotient rings as a finite dimensional representation of $\text{Aut } F_n$ in general. Moreover, he [12] also stated that in order to get accessible situation, it seems to be better to use rational functions rather than integral polynomials. In this paper, however, we consider the rational polynomials to obtain finite dimensional representations of $\text{Aut } F_n$.

The first purpose of the paper is to determine the structure of $\text{gr}^k(J)$ for $G = F_n$, $n \geq 3$ and $k = 1, 2$. Set

$$T := \{t'_i \mid 1 \leq i \leq n\} \cup \{t'_{ij} \mid 1 \leq i < j \leq n\} \cup \{t'_{ijk} \mid 1 \leq i < j < k \leq n\} \subset J$$

and

$$\begin{aligned} S := & \{t'_i t'_j \mid 1 \leq i \leq j \leq n\} \cup \{t'_i t'_{ab} \mid 1 \leq i \leq n, 1 \leq a < b \leq n\} \\ & \cup \{t'_i t'_{abc} \mid 1 \leq i \leq n, 1 \leq a < b < c \leq n\} \\ & \cup \{t'_{ij} t'_{ab} \mid 1 \leq i < j \leq n, 1 \leq a < b \leq n, (i, j) \leq (a, b)\}, \\ & \cup \{t'_{ab} t'_{abc}, t'_{ac} t'_{abc}, t'_{bc} t'_{abc} \mid 1 \leq a < b < c \leq n\} \\ & \cup \{t'_{ia} t'_{abc}, t'_{ib} t'_{abc}, t'_{ic} t'_{abc}, t'_{ia} t'_{ibc}, t'_{ab} t'_{iac}, t'_{ab} t'_{ibc}, t'_{ac} t'_{ibc}, t'_{ib} t'_{iac} \mid 1 \leq i < a < b < c \leq n\} \\ & \cup \{t'_{ja} t'_{ibc}, t'_{jb} t'_{iac}, t'_{jc} t'_{iab}, t'_{ab} t'_{ijc}, t'_{ac} t'_{ijb}, t'_{bc} t'_{ija} \mid 1 \leq i < j < a < b < c \leq n\} \\ & \subset J^2 \end{aligned}$$

respectively. We show

Theorem 1. (= Propositions 4.11, 4.12 and 4.13.) *For $G = F_n$ and $n \geq 3$, the sets T and S are basis of the \mathbf{Q} -vector spaces $\text{gr}^1(J)$ and $\text{gr}^2(J)$ respectively.*

In general, it seems to be very complicated to find a basis of $\text{gr}^k(J)$ for general $k \geq 3$.

Next, for any group G , we consider a descending filtration of $\text{Aut } G$. For any $k \geq 1$, let $\mathcal{E}_G(k)$ be the subgroup of $\text{Aut } G$ consisting of automorphisms which act on J/J^{k+1} trivially. Then we see that the groups $\mathcal{E}_G(k)$ define a descending filtration

$$\mathcal{E}_G(1) \supset \mathcal{E}_G(2) \supset \cdots \supset \mathcal{E}_G(k) \supset \cdots$$

of $\text{Aut } G$.

This filtration is a Fricke character analogue of the Andreadakis-Johnson filtration $\mathcal{A}_G(k)$ of $\text{Aut } G$. The Andreadakis-Johnson filtration was originally introduced by Andreadakis [2] in 1960's. In a series of his pioneer works [7], [8], [9] and [10], Johnson established the theory of Johnson homomorphisms in the study of the mapping class of surfaces. Together with the theory of the Johnson homomorphisms, the Andreadakis-Johnson filtration is one of powerful tools to study the group structure of the automorphism group of a group. (See Section 3 for notation, and see [14] or [15] for basic materials concerning the Andreadakis-Johnson filtration and the Johnson homomorphisms.)

The main purpose of the paper is to show

Proposition 1. (*= Proposition 5.3.*) For any $k, l \geq 1$, $[\mathcal{E}_G(k), \mathcal{E}_G(l)] \subset \mathcal{E}_G(k + l)$.

and

Theorem 2. (*= Theorems 5.12 and 5.13.*) For any $n \geq 3$,

- (1) $\mathcal{E}_{F_n}(1) = \text{Inn } F_n \cdot \mathcal{A}_{F_n}(2)$.
- (2) $\mathcal{A}_{F_n}(2k) \subset \mathcal{E}_{F_n}(k)$.

From Proposition 1, we see that $\{\mathcal{E}_G(k)\}$ is a central filtration of $\mathcal{E}_G(1)$. Then a natural problem to consider is how different is $\{\mathcal{E}_G(k)\}$ from the Andreadakis-Johnson filtration $\{\mathcal{A}_G(k)\}$. The partial answer to this question for $G = F_n$ is the theorem above.

On the other hand, since $\{\mathcal{E}_G(k)\}$ is central, each of the graded quotient $\text{gr}^k(\mathcal{E}_{F_n}) := \mathcal{E}_G(k)/\mathcal{E}_G(k + 1)$ is an abelian group. At the end of the paper, we show

Theorem 3. (*= Theorem 5.15.*) For any $n \geq 3$,

- (1) Each of $\text{gr}^k(\mathcal{E}_{F_n})$ is torsion-free.
- (2) $\dim_{\mathbf{Q}}(\text{gr}^k(\mathcal{E}_{F_n}) \otimes_{\mathbf{Z}} \mathbf{Q}) < \infty$.

To show this, we introduce Johnson homomorphism like homomorphisms η_k . Observing Theorem 2, we see that $\text{gr}^1(\mathcal{E}_{F_n})$ is finitely generated. In general, however, it seems to be quite a difficult to determine the structure of $\text{gr}^k(\mathcal{E}_{F_n})$ even the case where $k = 1$.

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2. NOTATION AND CONVENTIONS

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G .

- The abelianization of G is denoted by G^{ab} .

- The group $\text{Aut } G$ of G acts on G from the right. For any $\sigma \in \text{Aut } G$ and $x \in G$, the action of σ on x is denoted by x^σ .
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For elements x and y of G , the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

For pairs (i_1, i_2, \dots, i_k) and (j_1, j_2, \dots, j_k) of natural numbers $i_r, j_s \in \mathbf{N}$, we denote the lexicographic order among them by $(i_1, i_2, \dots, i_k) \leq (j_1, j_2, \dots, j_k)$. Namely, this means $i_1 < j_1$, $i_1 = j_1$ and $i_2 < j_2$, or and so on.

3. THE ANDREADAKIS-JOHNSON FILTRATION OF $\text{Aut } G$

In this section, we review the Andreadakis-Johnson filtration of $\text{Aut } G$ without proofs. The main purpose of the section is to fix the notations. For basic materials concerning the Andreadakis-Johnson filtration and the Johnson homomorphisms, see [14] or [15], for example.

For a group G , we define the lower central series of G by the rule

$$\Gamma_G(1) := F_n, \quad \Gamma_G(k) := [\Gamma_G(k-1), G], \quad k \geq 2.$$

For any $y_1, \dots, y_k \in G$, a left-normed commutator

$$[[\dots [[y_1, y_2], y_3], \dots], y_k]$$

of weight k is denoted by

$$[y_1, y_2, \dots, y_k]$$

for simplicity. Then we have

Lemma 3.1 (See Section 5.3 in [13]). *For any $k \geq 1$, the group $\Gamma_G(k)$ is generated by all left-normed commutators of weight k .*

Let $\rho_G : \text{Aut } G \rightarrow \text{Aut } G^{\text{ab}}$ be the natural homomorphism induced from the abelianization of G . The kernel $\text{IA}(G)$ of ρ_G is called the IA-automorphism group of G . Similarly, for any $k \geq 1$, the action of $\text{Aut } G$ on each nilpotent quotient group $G/\Gamma_G(k+1)$ induces a homomorphism

$$\rho_G^k : \text{Aut } G \rightarrow \text{Aut}(G/\Gamma_G(k+1)).$$

We denote the kernel of ρ_G^k by $\mathcal{A}_G(k)$. Then the groups $\mathcal{A}_G(k)$ define a descending central filtration

$$\text{IA}(G) = \mathcal{A}_G(1) \supset \mathcal{A}_G(2) \supset \dots$$

of $\text{Aut } G$. We call it the Andreadakis-Johnson filtration of $\text{Aut } G$. Then we have

Theorem 3.2 (Andreadakis [2]). *For any $k, l \geq 1$, $[\mathcal{A}_G(k), \mathcal{A}_G(l)] \subset \mathcal{A}_G(k+l)$.*

4. THE RING OF FRICKE CHARACTERS

In this section, we review the ring of Fricke characters of a finitely generated group G . In particular, we introduce a descending filtration consisting of $\text{Aut } G$ -invariant ideals of the ring.

4.1. An $\text{Aut } G$ -invariant ideal J .

Let G be a group generated by elements x_1, \dots, x_n . We denote by

$$R(G) := \text{Hom}(G, \text{SL}(2, \mathbf{C}))$$

the set of group homomorphisms from G to $\text{SL}(2, \mathbf{C})$. Let

$$\mathcal{F}(R(G), \mathbf{C}) := \{\chi : R(G) \rightarrow \mathbf{C}\}$$

be the set of complex-valued functions of $R(G)$. Then we can consider $\mathcal{F}(R(G), \mathbf{C})$ as a commutative ring in the following usual manner. For any χ and $\chi' \in \mathcal{F}(R(G), \mathbf{C})$, we define the sum and the product of χ and χ' to be

$$\begin{aligned} (\chi + \chi')(\rho) &:= \chi(\rho) + \chi'(\rho), \\ (\chi\chi')(\rho) &:= \chi(\rho)\chi'(\rho) \end{aligned}$$

for any $\rho \in R(G)$ respectively. We consider $R(G)$ and $\mathcal{F}(R(G), \mathbf{C})$ as right $\text{Aut}(G)$ -modules by

$$\rho^\sigma(x) := \rho(x^{\sigma^{-1}}), \quad \rho \in R(G) \text{ and } x \in G$$

and

$$\chi^\sigma(\rho) := \chi(\rho^{\sigma^{-1}}), \quad \chi \in \mathcal{F}(R(G), \mathbf{C}) \text{ and } \rho \in R(G)$$

respectively.

For any $x \in G$, we define an element $\text{tr } x$ of $\mathcal{F}(R(G), \mathbf{C})$ to be

$$(\text{tr } x)(\rho) := \text{tr } \rho(x)$$

for any $\rho \in R(G)$. Here “tr” in the right hand side means the trace of 2×2 matrix $\rho(x)$. The element $\text{tr } x$ in $\mathcal{F}(R(G), \mathbf{C})$ is called the Fricke character of $x \in G$. The action of an element $\sigma \in \text{Aut}(G)$ on $\text{tr } x$ is given by $\text{tr } x^\sigma$. We have the following well-known formulae:

- (1) $\text{tr } x^{-1} = \text{tr } x$,
- (2) $\text{tr } xy = \text{tr } yx$,
- (3) $\text{tr } xy + \text{tr } xy^{-1} = (\text{tr } x)(\text{tr } y)$,
- (4) $\text{tr } xyz + \text{tr } yxz = (\text{tr } x)(\text{tr } yz) + (\text{tr } y)(\text{tr } xz) + (\text{tr } z)(\text{tr } xy) - (\text{tr } x)(\text{tr } y)(\text{tr } z)$,
- (5) $\text{tr } [x, y] = (\text{tr } x)^2 + (\text{tr } y)^2 + (\text{tr } xy)^2 - (\text{tr } x)(\text{tr } y)(\text{tr } xy) - 2$,
- (6) $2\text{tr } xyzw = (\text{tr } x)(\text{tr } yzw) + (\text{tr } y)(\text{tr } zwx) + (\text{tr } z)(\text{tr } wxy) + (\text{tr } w)(\text{tr } xyz)$
 $+ (\text{tr } xy)(\text{tr } zw) - (\text{tr } xz)(\text{tr } yw) + (\text{tr } xw)(\text{tr } yz)$
 $- (\text{tr } x)(\text{tr } y)(\text{tr } zw) - (\text{tr } y)(\text{tr } z)(\text{tr } xw) - (\text{tr } x)(\text{tr } w)(\text{tr } yz)$
 $- (\text{tr } z)(\text{tr } w)(\text{tr } xy) + (\text{tr } x)(\text{tr } y)(\text{tr } z)(\text{tr } w)$

for any $x, y, z, w \in G$. The equations (4) and (6) are due to Vogt [16]. (For details, see Section 3.4 in [11] for example.)

Let $\mathfrak{X}(G)$ be the \mathbf{Z} -submodule of $\mathcal{F}(R(G), \mathbf{C})$ generated by all $\text{tr } x$ for $x \in G$. Then, from (3), it is seen that $\mathfrak{X}(G)$ is closed under the multiplication of $\mathcal{F}(R(G), \mathbf{C})$. Consider an integral polynomial ring

$$\mathbf{Z}[t] := \mathbf{Z}[t_{i_1 \dots i_l} \mid 1 \leq l \leq n, 1 \leq i_1 < i_2 < \dots < i_l \leq n]$$

of $2^n - 1$ indeterminates, and a ring homomorphism $\pi : \mathbf{Z}[t] \rightarrow \mathcal{F}(R(G), \mathbf{C})$ defined by

$$\pi(1) := \frac{1}{2}(\text{tr } 1_G), \quad \pi(t_{i_1 \dots i_l}) := \text{tr } x_{i_1} \cdots x_{i_l}.$$

Fricke [3] stated that for any element x which is a word $x \in G$ in the generators x_1, \dots, x_n , the character $\text{tr } x$ is a polynomial among $\text{tr } x_{i_1} x_{i_2} \cdots x_{i_l}$ for $1 \leq l \leq n$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. This was proved by Horowitz [5]. More precisely,

Theorem 4.1 (Horowitz, [5]). *For any G , $\mathfrak{X}(G)$ is the image of an ideal*

$$I_0 := (2, t_{i_1 \dots i_l} \mid 1 \leq l \leq n, 1 \leq i_1 < i_2 < \cdots < i_l \leq n) \subset \mathbf{Z}[t]$$

by π .

Set $I := \text{Ker}(\pi)$. Namely,

$$I = \{f \in \mathbf{Z}[t] \mid f(\text{tr } \rho(x_{i_1} \cdots x_{i_l})) = 0 \text{ for any } \rho \in R(G)\}.$$

Horowitz [5] also showed that $I = (0)$ for $n = 1$ and 2, and that for $n = 3$ and $G = F_3$, I is a principal ideal generated by a quadratic element

$$t_{123}^2 - P_{123}(t)t_{123} + Q_{123}(t)$$

where

$$P_{abc}(t) := t_{ab}t_c + t_{ac}t_b + t_{bc}t_a,$$

$$Q_{abc}(t) := t_a^2 + t_b^2 + t_c^2 + t_{ab}^2 + t_{ac}^2 + t_{bc}^2 - t_a t_b t_{ab} - t_a t_c t_{ac} - t_b t_c t_{bc} + t_{ab} t_{bc} t_{ac} - 4.$$

For $n \geq 4$ and $G = F_n$, Whittlemore [17] showed that I is not principal. In general, however, very little is known for the ideal I for general $n \geq 4$.

In this paper, we call the quotient ring $\mathbf{Z}[t]/I$ the ring of Fricke characters of G over \mathbf{Z} , and considered as a subring of $\mathcal{F}(R(G), \mathbf{C})$ through the homomorphism π . Then, we can define an $\text{Aut}(G)$ -module structure of $\mathbf{Z}[t]/I$ such that the induced homomorphism $\mathbf{Z}[t]/I \rightarrow \mathcal{F}(R(G), \mathbf{C})$ from π is $\text{Aut}(G)$ -equivariant injective.

For an elements $y \in G$, an automorphism ι_y of G defined by $x^{\iota_y} := y^{-1}xy$ for any $x \in G$ is called an inner automorphism of G associated to y . Let $\text{Inn}(G)$ be a normal subgroup of $\text{Aut}(G)$ consisting of inner automorphisms of G . In general, $\text{Inn}(G)$ is contained in the kernel of a homomorphism $\zeta : \text{Aut } G \rightarrow \text{Aut}(\mathbf{Z}[t]/I)$ induced from the action of $\text{Aut } G$ on the ring of Fricke characters. For the case where G is a free group $F_n = \langle x_1, \dots, x_n \rangle$ of rank n , Horowitz [6] showed

Theorem 4.2 (Horowitz, [6]). *For $n \geq 3$, $\text{Ker}(\zeta) = \text{Inn } F_n$.*

Namely, the action of $\text{Aut } F_n$ on the ring of Fricke characters induces a faithful representation of the outer automorphism group $\text{Out } F_n := \text{Aut } F_n / \text{Inn } F_n$ of a free group F_n . However, since $\mathbf{Z}[t]/I$ is not finitely generated as a \mathbf{Z} -module, the representations $\text{Aut } F_n \rightarrow \text{Aut}(\mathbf{Z}[t]/I)$ and $\text{Out } F_n \rightarrow \text{Aut}(\mathbf{Z}[t]/I)$ are not so easy to handle in general. In addition to this, the number of indeterminates of $\mathbf{Z}[t]$ also adds momentum to the complexity if we write down the behavior of the actions of $\text{Aut } F_n$ and $\text{Out } F_n$ on $\mathbf{Z}[t]/I$.

In order to avoid these difficulties, first, we consider the rationalization of the situation above. Let $\mathfrak{X}_{\mathbf{Q}}(G)$ be a \mathbf{Q} -subspace of $\mathcal{F}(R(G), \mathbf{C})$ generated by all $\text{tr } x$ for $x \in G$. The set $\mathfrak{X}_{\mathbf{Q}}(G)$ naturally has a ring structure. Let

$$\mathbf{Q}[t] := \mathbf{Q}[t_{i_1 \dots i_l} \mid 1 \leq l \leq 3, 1 \leq i_1 < i_2 < \cdots < i_l \leq n]$$

be a rational polynomial ring of $n + \binom{n}{2} + \binom{n}{3}$ indeterminates. Consider a ring homomorphism $\pi_{\mathbf{Q}} : \mathbf{Q}[t] \rightarrow \mathcal{F}(R(G), \mathbf{C})$ defined by

$$\pi_{\mathbf{Q}}(1) := \frac{1}{2}(\text{tr } 1_G), \quad \pi_{\mathbf{Q}}(t_{i_1 \dots i_l}) := \text{tr } x_{i_1} \cdots x_{i_l}.$$

Then observing the formula (6) and Horowitz's result as mentioned above, we see $\text{Im}(\pi_{\mathbf{Q}}) = \mathfrak{X}_{\mathbf{Q}}(G)$. Remark that $\text{Im}(\pi) \neq \mathfrak{X}(G)$. Set

$$I_{\mathbf{Q}} := \text{Ker}(\pi_{\mathbf{Q}}) = \{f \in \mathbf{Q}[t] \mid f(\text{tr } \rho(x_{i_1} \cdots x_{i_l})) = 0 \text{ for any } \rho \in R(G)\}.$$

We call $\mathfrak{X}_{\mathbf{Q}}(G)$ and $\mathbf{Q}[t]/I_{\mathbf{Q}}$ the ring of Fricke characters of G over \mathbf{Q} . Similar to $\mathbf{Z}[t]/I$, we see that $\mathbf{Q}[t]/I_{\mathbf{Q}}$ can be considered as an $\text{Aut } G$ -module, and that $\text{Inn } G$ is contained in the kernel of a homomorphism $\zeta_{\mathbf{Q}} : \text{Aut } G \rightarrow \text{Aut}(\mathbf{Q}[t]/I)$ induced from the action of $\text{Aut } G$ on $\mathbf{Q}[t]/I$.

If $G = F_n$, since $\text{Ker}(\zeta_{\mathbf{Q}})$ acts on $\mathfrak{X}(F_n) \subset \mathfrak{X}_{\mathbf{Q}}(F_n)$ trivially, we see that $\text{Ker}(\zeta_{\mathbf{Q}}) = \text{Inn } F_n$ by Theorem 4.2. Hence, $\zeta_{\mathbf{Q}}$ also induces a faithful representation of $\text{Out } F_n$. In order to construct finite dimensional representations of $\text{Aut } G$ and $\text{Out } G$, we prepare a descending filtration of $\text{Aut } G$ -invariant ideals of $\mathbf{Q}[t]/I_{\mathbf{Q}}$, and take its graded quotients. Set $t'_{i_1 \dots i_l} := t_{i_1 \dots i_l} - 2 \in \mathbf{Q}[t]$. We also denote by $t'_{i_1 \dots i_l}$ its coset class in $\mathbf{Q}[t]/I_{\mathbf{Q}}$. Consider an ideal

$$J := (t'_{i_1 \dots i_l} \mid 1 \leq l \leq 3, 1 \leq i_1 < i_2 < \cdots < i_l \leq n) \subset \mathbf{Q}[t]/I_{\mathbf{Q}}$$

generated by all $t'_{i_1 \dots i_l}$'s.

Lemma 4.3 (For $n = 3$, see also Magnus [12]). *The ideal J is $\text{Aut } G$ -invariant.*

Proof. For any $t'_{j_1 \dots j_m}$ and $\sigma \in \text{Aut } G$, there exists some polynomial $F(t_{i_1 \dots i_l}) \in \mathbf{Q}[t]$ such that

$$(t'_{j_1 \dots j_m})^{\sigma} \equiv F(t_{i_1 \dots i_l}) \in \mathbf{Q}[t]/I_{\mathbf{Q}}$$

by Theorem 4.1 and (6). Then using the division algorithm, we verify that F can be written as

$$F = t'_1 G_1 + R \in \mathbf{Q}[t]$$

where $G_1, R \in \mathbf{Q}[t]$ such that R is a polynomial in the determinates $t'_{i_1 \dots i_l}$ except for t'_1 . By repeating this argument, we obtain

$$F = \sum_{i_1 < \cdots < i_l} t'_{i_1 \dots i_l} G_{i_1 \dots i_l} + C$$

where $G_{i_1 \dots i_l} \in \mathbf{Q}[t]$, $C \in \mathbf{Q}$, and the sum runs over all $i_1 < \cdots < i_l$ such that $1 \leq l \leq 3$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$.

By considering the image of this equation by $\zeta_{\mathbf{Q}}$, and by substituting the trivial representation $\mathbf{1} : R(G) \rightarrow \text{SL}(2, \mathbf{C})$, we see that $C = 0$. This means $F \in J$. Therefore, J is $\text{Aut } G$ -invariant. \square

Now, we have a descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots$$

of $\text{Aut } G$ -invariant ideals of $\mathbf{Q}[t]/I_{\mathbf{Q}}$. Set

$$\text{gr}^k(J) := J^k / J^{k+1}.$$

Then each of $\text{gr}^k(J)$ is an $\text{Aut } G$ -invariant \mathbf{Q} -vector space of finite dimension for any $k \geq 1$. Hence, we obtain finite dimensional representations

$$\zeta_{k,\mathbf{Q}} : \text{Aut } G \rightarrow \text{Aut}(\text{gr}^k(J))$$

over \mathbf{Q} for $k \geq 1$. In general, by combinatorial complexities, it seems quite a difficult to give a basis of $\text{gr}^k(J)$ and to give a basis of it even in the case where $G = F_n$. In the present paper, we determine the \mathbf{Q} -vector space structures of $\text{gr}^k(J)$ for $G = F_n$ and $k = 1, 2$ in Subsection 4.3.

4.2. Basic formulae among $\text{tr}' x$.

For any $x \in G$, set

$$\text{tr}' x := (\text{tr } x) - 2 \in \mathcal{F}(R(G), \mathbf{C}).$$

In this subsection, we summarize basic and useful formulae among $\text{tr}' x$. To begin with, in order to rewrite the equations (1), ..., (6) as those among $\text{tr}' x, \dots, \text{tr}' w$, we prepare

Lemma 4.4. *For any $k \geq 1$, and $z_1, \dots, z_k \in G$, we have*

$$(\text{tr } z_1) \cdots (\text{tr } z_k) = \sum_{i=0}^k 2^i \sum_{1 \leq j_1 < \cdots < j_{k-i} \leq k} (\text{tr}' z_{j_1}) \cdots (\text{tr}' z_{j_{k-i}})$$

Since this formula can be shown easily with the induction on k , we omit the details. \square

Then using the lemma above and (1), ..., (6), we obtain

$$(7) \quad \text{tr}' x^{-1} = \text{tr}' x,$$

$$(8) \quad \text{tr}' xy = \text{tr}' yx,$$

$$(9) \quad \text{tr}' xy + \text{tr}' xy^{-1} = 2\text{tr}' x + 2\text{tr}' y + (\text{tr}' x)(\text{tr}' y),$$

$$(10) \quad \begin{aligned} \text{tr}' xyz + \text{tr}' yxz = & -2\{\text{tr}' x + \text{tr}' y + \text{tr}' z\} + 2\{\text{tr}' xy + \text{tr}' yz + \text{tr}' xz\} \\ & + (\text{tr}' x)(\text{tr}' yz) + (\text{tr}' y)(\text{tr}' xz) + (\text{tr}' z)(\text{tr}' xy), \\ & -2\{(\text{tr}' x)(\text{tr}' y) + (\text{tr}' y)(\text{tr}' z) + (\text{tr}' z)(\text{tr}' x)\} \\ & - (\text{tr}' x)(\text{tr}' y)(\text{tr}' z), \end{aligned}$$

$$(11) \quad \begin{aligned} \text{tr}' [x, y] = & (\text{tr}' x)^2 + (\text{tr}' y)^2 + (\text{tr}' xy)^2 \\ & - 2\{(\text{tr}' x)(\text{tr}' y) + (\text{tr}' x)(\text{tr}' xy) + (\text{tr}' y)(\text{tr}' xy)\} - (\text{tr}' x)(\text{tr}' y)(\text{tr}' xy) \end{aligned}$$

and

$$\begin{aligned}
(12) \quad 2\text{tr}' xyzw &= 2(\text{tr}' x + \text{tr}' y + \text{tr}' z + \text{tr}' w) \\
&\quad - 2(\text{tr}' xy + \text{tr}' xz + \text{tr}' xw + \text{tr}' yz + \text{tr}' yw + \text{tr}' zw) \\
&\quad + 2(\text{tr}' xyz + \text{tr}' xyw + \text{tr}' xzw + \text{tr}' yzw) \\
&\quad + 2\{(\text{tr}' x)(\text{tr}' y) + (\text{tr}' x)(\text{tr}' w) + (\text{tr}' y)(\text{tr}' z) + (\text{tr}' z)(\text{tr}' w) \\
&\quad \quad + 2(\text{tr}' x)(\text{tr}' z) + 2(\text{tr}' y)(\text{tr}' w)\} \\
&\quad - 2\{(\text{tr}' x)(\text{tr}' yz) + (\text{tr}' x)(\text{tr}' zw) + (\text{tr}' y)(\text{tr}' xw) + (\text{tr}' y)(\text{tr}' zw) \\
&\quad \quad + (\text{tr}' z)(\text{tr}' xy) + (\text{tr}' z)(\text{tr}' xw) + (\text{tr}' w)(\text{tr}' xy) + (\text{tr}' w)(\text{tr}' yz)\} \\
&\quad + \{(\text{tr}' x)(\text{tr}' yzw) + (\text{tr}' y)(\text{tr}' xzw) + (\text{tr}' z)(\text{tr}' xyw) + (\text{tr}' w)(\text{tr}' xyz)\} \\
&\quad + \{(\text{tr}' xy)(\text{tr}' zw) - (\text{tr}' xz)(\text{tr}' yw) + (\text{tr}' xw)(\text{tr}' yz)\} \\
&\quad - \{(\text{tr}' x)(\text{tr}' y)(\text{tr}' zw) + (\text{tr}' y)(\text{tr}' z)(\text{tr}' xw) + (\text{tr}' x)(\text{tr}' w)(\text{tr}' yz) \\
&\quad \quad + (\text{tr}' z)(\text{tr}' w)(\text{tr}' xy)\} \\
&\quad + (\text{tr}' x)(\text{tr}' y)(\text{tr}' z)(\text{tr}' w) \\
&\quad + 2\{(\text{tr}' x)(\text{tr}' y)(\text{tr}' z) + (\text{tr}' x)(\text{tr}' y)(\text{tr}' w) + (\text{tr}' x)(\text{tr}' z)(\text{tr}' w) \\
&\quad \quad + (\text{tr}' y)(\text{tr}' z)(\text{tr}' w)\}.
\end{aligned}$$

Furthermore, we can rewrite (12) as a sum of the degree one part and elements types of

$$(\text{tr}' \alpha)(\text{tr}' \beta w - \text{tr}' \beta), \quad (\text{tr}' \alpha)(\text{tr}' w), \quad (\text{tr}' \alpha)(\text{tr}' \beta)(\text{tr}' w)$$

for some $\alpha, \beta \in G$ and

$$(\text{tr}' x)(\text{tr}' y)(\text{tr}' z)(\text{tr}' w).$$

That is,

$$\begin{aligned}
(13) \quad 2\text{tr}' xyzw &= 2(\text{tr}' x + \text{tr}' y + \text{tr}' z + \text{tr}' w) \\
&\quad - 2(\text{tr}' xy + \text{tr}' xz + \text{tr}' xw + \text{tr}' yz + \text{tr}' yw + \text{tr}' zw) \\
&\quad + 2(\text{tr}' xyz + \text{tr}' xyw + \text{tr}' xzw + \text{tr}' yzw) \\
&\quad + 2\{(\text{tr}' x - \text{tr}' xw)(\text{tr}' y) + (\text{tr}' y)(\text{tr}' z - \text{tr}' zw) + (\text{tr}' x - \text{tr}' xw)(\text{tr}' z) \\
&\quad \quad + (\text{tr}' x)(\text{tr}' z - \text{tr}' zw)\} \\
&\quad - (\text{tr}' x)(\text{tr}' yz - \text{tr}' yzw) - (\text{tr}' x - \text{tr}' xw)(\text{tr}' yz) - (\text{tr}' z - \text{tr}' zw)(\text{tr}' xy) \\
&\quad - (\text{tr}' z)(\text{tr}' xy - \text{tr}' xyw) + (\text{tr}' y)(\text{tr}' xzw - \text{tr}' xz) + (\text{tr}' xz)(\text{tr}' y - \text{tr}' yw) \\
&\quad + (\text{tr}' x)(\text{tr}' y)(\text{tr}' z - \text{tr}' zw) + (\text{tr}' y)(\text{tr}' z)(\text{tr}' x - \text{tr}' xw) \\
&\quad + 2(\text{tr}' x)(\text{tr}' w) + 2(\text{tr}' z)(\text{tr}' w) + 4(\text{tr}' y)(\text{tr}' w) - 2(\text{tr}' w)(\text{tr}' xy) \\
&\quad - 2(\text{tr}' w)(\text{tr}' yz) + (\text{tr}' w)(\text{tr}' xyz) \\
&\quad + (\text{tr}' x)(\text{tr}' y)(\text{tr}' z)(\text{tr}' w) + (\text{tr}' x)(\text{tr}' w)(\text{tr}' yz) + (\text{tr}' z)(\text{tr}' w)(\text{tr}' xy) \\
&\quad + 2\{(\text{tr}' x)(\text{tr}' y)(\text{tr}' w) + (\text{tr}' x)(\text{tr}' z)(\text{tr}' w) + (\text{tr}' y)(\text{tr}' z)(\text{tr}' w)\}.
\end{aligned}$$

Next, we consider elements type of $\text{tr}' z\gamma$ for $z \in G$ and $\gamma \in [G, G]$. First, we study the case where γ is a commutator of weight 2.

Lemma 4.5. *For any $z, a, b \in G$,*

$$\begin{aligned}
\text{tr}'z[a, b] &= \text{tr}'z + 2(\text{tr}'z + \text{tr}'a + \text{tr}'b) \\
&\quad - 2(\text{tr}'za + \text{tr}'zb + \text{tr}'ab) + 2\text{tr}'zab \\
&\quad + (\text{tr}'za)(\text{tr}'a) - (\text{tr}'zb)(\text{tr}'b) + 4(\text{tr}'z)(\text{tr}'b) + 2(\text{tr}'b)^2 \\
&\quad - 2(\text{tr}'za)(\text{tr}'b) - 2(\text{tr}'ab)(\text{tr}'b) - 2(\text{tr}'za)(\text{tr}'ab) \\
&\quad + (\text{tr}'ab)(\text{tr}'zab) + (\text{tr}'z)(\text{tr}'b)^2 - (\text{tr}'za)(\text{tr}'ab)(\text{tr}'b) \\
&= \text{tr}'z - 2(\text{tr}'za - \text{tr}'z) + 2(\text{tr}'bza - \text{tr}'bz) - 2(\text{tr}'ba - \text{tr}'b) \\
&\quad + (\text{tr}'bza - \text{tr}'bz)(\text{tr}'b) + (\text{tr}'ba - \text{tr}'b)(\text{tr}'bza) - 2(\text{tr}'za - \text{tr}'z)(\text{tr}'b) \\
&\quad - 2(\text{tr}'ba - \text{tr}'b)(\text{tr}'b) - 2(\text{tr}'za - \text{tr}'z)(\text{tr}'b) - 2(\text{tr}'ba - \text{tr}'b)(\text{tr}'za) \\
&\quad - (\text{tr}'za - \text{tr}'z)(\text{tr}'b)^2 - (\text{tr}'ba - \text{tr}'b)(\text{tr}'b)(\text{tr}'za) \\
&\quad + 2\text{tr}'a + (\text{tr}'za)(\text{tr}'a).
\end{aligned}$$

Proof. We show the former equality. The latter one immediately follows from the former one. Now, we have

$$\begin{aligned}
\text{tr}'z[a, b] &= \text{tr}'zab(ba)^{-1} \\
&\stackrel{(9)}{=} -\text{tr}'zab^2a + (\text{tr}'zab)(\text{tr}'ab) + 2\text{tr}'zab + 2\text{tr}'ab \\
&= -\text{tr}'(azab)b + (\text{tr}'zab)(\text{tr}'ab) + 2\text{tr}'zab + 2\text{tr}'ab \\
&\stackrel{(9)}{=} \text{tr}'aza - (\text{tr}'azab)(\text{tr}'b) - 2\text{tr}'azab - 2\text{tr}'b \\
&\quad + (\text{tr}'zab)(\text{tr}'ab) + 2\text{tr}'zab + 2\text{tr}'ab \\
&= \text{tr}'(za)a - (\text{tr}'azab)(\text{tr}'b) - 2\text{tr}'azab - 2\text{tr}'b \\
&\quad + (\text{tr}'zab)(\text{tr}'ab) + 2\text{tr}'zab + 2\text{tr}'ab \\
&\stackrel{(9)}{=} -\text{tr}'z + (\text{tr}'za)(\text{tr}'a) + 2\text{tr}'za + 2\text{tr}'a \\
&\quad - \{-\text{tr}'azb^{-1}a^{-1} + (\text{tr}'za)(\text{tr}'ab) + 2\text{tr}'za + 2\text{tr}'ab\}(\text{tr}'b) \\
&\quad - 2\{-\text{tr}'azb^{-1}a^{-1} + (\text{tr}'za)(\text{tr}'ab) + 2\text{tr}'za + 2\text{tr}'ab\} \\
&\quad - 2\text{tr}'b + (\text{tr}'zab)(\text{tr}'ab) + 2\text{tr}'zab + 2\text{tr}'ab.
\end{aligned}$$

Using this and

$$\text{tr}'zb^{-1} \stackrel{(9)}{=} -\text{tr}'zb + (\text{tr}'z)(\text{tr}'b) + 2\text{tr}'z + 2\text{tr}'b,$$

we obtain the required result. This completes the proof of Lemma 4.5. \square

Using Lemma 4.5 and (10), we see

Corollary 4.6. *For any $z, a, b \in G$, $\text{tr}'z[a, b] \equiv \text{tr}'z + \text{tr}'zab - \text{tr}'zba \pmod{J^2}$.*

Now, we consider the case where γ is a commutator of weight 3.

Lemma 4.7. *For any $z, a, b \in G$, an element $\text{tr}'z[a, b, c] - \text{tr}'z$ is a sum of elements types of*

$$(\text{tr}'x)(\text{tr}'y[a, b] - \text{tr}'y), \quad (\text{tr}'x)(\text{tr}'ya - \text{tr}'y)$$

and

$$\text{tr}'[a, b], \quad (\text{tr}'x)(\text{tr}'[a, b]), \quad (\text{tr}'x)(\text{tr}'a)$$

for some $x, y \in G$.

Proof. By substituting a and b in the equation in Lemma 4.5 to $[a, b]$ and c respectively, we obtain

$$\begin{aligned} \text{tr}'z[a, b, c] &= \text{tr}'z - 2(\text{tr}'z[a, b] - \text{tr}'z) + 2(\text{tr}'cz[a, b] - \text{tr}'cz) - 2(\text{tr}'c[a, b] - \text{tr}'c) \\ &\quad + 2\text{tr}'[a, b] + (\text{tr}'z[a, b])(\text{tr}'[a, b]) + \sum (\text{tr}'x)(\text{tr}'y[a, b] - \text{tr}'y). \end{aligned}$$

Again, by Lemma 4.5, we have

$$\begin{aligned} \text{tr}'z[a, b, c] &= \text{tr}'z - 2\{-2(\text{tr}'za - \text{tr}'z) + 2(\text{tr}'bza - \text{tr}'bz) - 2(\text{tr}'ba - \text{tr}'b)\} \\ &\quad + 2\{-2(\text{tr}'cza - \text{tr}'cz) + 2(\text{tr}'bcza - \text{tr}'bcz) - 2(\text{tr}'ba - \text{tr}'b)\} \\ &\quad - 2\{-2(\text{tr}'ca - \text{tr}'c) + 2(\text{tr}'bca - \text{tr}'bc) - 2(\text{tr}'ba - \text{tr}'b)\} \\ &\quad - 4\text{tr}'a + \sum (\text{tr}'x)(\text{tr}'a) + \sum (\text{tr}'x)(\text{tr}'ya - \text{tr}'y) \\ &\quad + 2\text{tr}'[a, b] + (\text{tr}'z[a, b])(\text{tr}'[a, b]) + \sum (\text{tr}'x)(\text{tr}'y[a, b] - \text{tr}'y). \end{aligned}$$

Then by using (13) for $\text{tr}'bcza$, we obtain the required result. This completes the proof of Lemma 4.7. \square

In particular, we have

Corollary 4.8. $\text{tr}'z[a, b, c]^{\pm 1} \equiv \text{tr}'z \pmod{J^2}$.

Proof. For $\text{tr}'z[a, b, c]^{-1} \equiv \text{tr}'z \pmod{J^2}$, it suffices to consider

$$\text{tr}'z[a, b, c]^{-1} = \text{tr}'z[cac^{-1}, cbc^{-1}, c^{-1}].$$

This completes the proof of Corollary 4.8. \square

The next proposition is the goal of this subsection. For any $k \geq 2$ and $y_1, \dots, y_k \in G$, set

$$a_k := [y_1, y_2, \dots, y_k] \in \Gamma_G(k).$$

Proposition 4.9. *With the notation above, we have*

- (1) For any $k \geq 2$, $\text{tr}'a_k \in J^{k-1}$.
- (2) For any $l \geq 2$ and any $b \in G$,

$$\text{tr}'ba_{2l-1} - \text{tr}'b, \quad \text{tr}'ba_{2l} - \text{tr}'b \in J^l.$$

Proof. We prove this proposition by the induction on k and l . To begin with, for $k = l = 2$, the part (1) and (2) follows from (11) and Corollary 4.8 respectively. Furthermore, the part (1) also holds for $k = 3$ by (11).

Assume $l \geq 2$ and $k = 2l - 2 \geq 2$, and assume that the part (1) is true for k and $k + 1$, and that the part (2) is true for l . We show that for any k' such that $k + 2 \leq k' \leq k + 3$,

the part (1) holds. By the inductive hypothesis, we have

$$\begin{aligned}
\text{tr}' a_{k'} &= \text{tr}' [a_{k'-1}, y_{k'}] \\
&\stackrel{(11)}{=} (\text{tr}' a_{k'-1})^2 + (\text{tr}' y_{k'})^2 + (\text{tr}' a_{k'-1} y_{k'})^2 \\
&\quad - 2\{(\text{tr}' a_{k'-1})(\text{tr}' y_{k'}) + (\text{tr}' a_{k'-1})(\text{tr}' a_{k'-1} y_{k'}) + (\text{tr}' y_{k'})(\text{tr}' a_{k'-1} y_{k'})\} \\
&\quad - (\text{tr}' a_{k'-1})(\text{tr}' y_{k'})(\text{tr}' a_{k'-1} y_{k'}) \\
&\equiv (\text{tr}' a_{k'-1} y_{k'} - \text{tr}' y_{k'})^2 \pmod{J^{k'-1}} \\
&\equiv 0 \pmod{J^{k'-1}}.
\end{aligned}$$

This shows that for $k' = k + 1, k + 2$, the part (1) holds.

Next, for $l' = l + 1$, by Lemma 4.7, we have

$$\begin{aligned}
&\text{tr}' ba_{2l'-1} - \text{tr}' b \\
&= \sum (\text{tr}' x)(\text{tr}' ya_{2l'-3} - \text{tr}' y) + \sum (\text{tr}' x)(\text{tr}' ya_{2l'-2} - \text{tr}' y) \\
&\quad + m\text{tr}' a_{2l'-2} + \sum (\text{tr}' x)(\text{tr}' a_{2l'-2}) + \sum (\text{tr}' x)(\text{tr}' a_{2l'-3}) \\
&\equiv 0 \pmod{J^{l'}}
\end{aligned}$$

for some $m \in \mathbf{Z}$. Here $\text{tr}' a_{2l'-2} = \text{tr}' a_{2l} \in J^{2l-1} \subset J^{l'}$ from the argument above. Therefore we see that the part (2) holds for $l' = l + 1$. This completes the proof of Proposition 4.9. \square

Corollary 4.10. *With the notation above, we have*

- (1) For any $k \geq 2$, $\text{tr}' a_k^{-1} \in J^{k-1}$.
- (2) For any $l \geq 2$ and $b \in G$,

$$\text{tr}' ba_{2l-1}^{-1} - \text{tr}' b, \text{tr}' ba_{2l}^{-1} - \text{tr}' b \in J^l.$$

Proof. This corollary is immediately proved by Proposition 4.9 and

$$[\alpha, \beta]^{-1} = [\beta\alpha\beta^{-1}, \beta^{-1}]$$

for any $\alpha, \beta \in G$. This completes the proof of Corollary 4.10. \square

4.3. The structures of $\text{gr}^k(J)$ for $G = F_n$ and $k = 1$ and 2.

In this subsection, we assume $G = F_n$. The goal of this subsection is to give a basis of $\text{gr}^k(J)$ for $k = 1$ and 2.

4.3.1. A basis of $\text{gr}^1(J)$.

Here we show that the image of

$$T := \{t'_i \mid 1 \leq i \leq n\} \cup \{t'_{ij} \mid 1 \leq i < j \leq n\} \cup \{t'_{ijk} \mid 1 \leq i < j < k \leq n\} \subset \mathbf{Q}[t]$$

by $\pi_{\mathbf{Q}}$ forms a basis of $\text{gr}^1(J)$ as a \mathbf{Q} -vector space. To do this, it suffices to show

Proposition 4.11. *For an ideal $J_0 := (T)$ of $\mathbf{Q}[t]$ generated by T , we have $I_{\mathbf{Q}} \subset J_0^2$.*

Proof. For any $f \in I_{\mathbf{Q}}$, set

$$f := \sum_{1 \leq i \leq n} a_i t'_i + \sum_{1 \leq i < j \leq n} a_{ij} t'_{ij} + \sum_{i < j < k} a_{ijk} t'_{ijk} \\ + (\text{terms of degree } \geq 2)$$

for $a_i, a_{ij}, a_{ijk} \in \mathbf{Q}$. Then, it suffices to show that $a_i = a_{ij} = a_{ijk} = 0$.

Choose any $i < j < k$ and fix it. Consider the interior

$$D := \{z \in \mathbf{C} \mid z\bar{z} < 1\}$$

of the unit disk in \mathbf{C} . For any $s \in D$ and $l, m, t, u \in \mathbf{C}$, define a representation $\rho_1 : F_n \rightarrow \text{SL}(2, \mathbf{C})$ by

$$\rho_1(x_i) := \begin{pmatrix} 1-s & 0 \\ 0 & (1-s)^{-1} \end{pmatrix}, \quad \rho_1(x_j) := \begin{pmatrix} 1-lt & l^2t \\ -t & 1+lt \end{pmatrix}, \quad \rho_1(x_k) := \begin{pmatrix} 1-mu & m^2u \\ -u & 1+mu \end{pmatrix}.$$

If we consider the power series expansion

$$\frac{1}{1-s} = 1 + s + s^2 + s^3 + \dots$$

at the origin on D , we can write $\text{tr}' \rho_1(x_{i_1} \cdots x_{i_l})$ as a convergent power series of s, t, u :

$$\begin{aligned} \text{tr}' \rho_1(x_i) &= \frac{s^2}{1-s} = s^2 + s^3 + s^4 + \dots, \\ \text{tr}' \rho_1(x_i x_j) &= \frac{1}{1-s} (s^2 + 2lst) \\ &= s^2 + 2lst + s^3 + 2ls^2t + (\text{terms of degree } \geq 4), \\ \text{tr}' \rho_1(x_j x_k) &= -(l-m)^2 tu, \\ \text{tr}' \rho_1(x_i x_j x_k) &= \frac{1}{1-s} \{s^2 + 2lst + 2msu - (m-l)^2 tu + 2l(l-m)stu \\ &\quad - ls^2t - ms^2u + l(m-l)s^2tu\} \\ &= s^2 + s^3 + 2lst + 2msu - (m-l)^2 tu + (l^2 - m^2)stu + ls^2t + ms^2u \\ &\quad + (\text{terms of degree } \geq 4), \end{aligned}$$

and so on. This shows that $\text{tr}' \rho_1(x_{i_1} \cdots x_{i_l})$ is equal to zero, or the degrees of its monomials are greater than one. Then we have

$$\begin{aligned}
(14) \quad & f(\text{tr}' \rho_1(x_{i_1} \cdots x_{i_l})) \\
&= a_i(\text{tr}' \rho_1(x_i)) + a_{jk}(\text{tr}' \rho_1(x_j x_k)) + \sum_{r < i} a_{ri}(\text{tr}' \rho_1(x_r x_i)) + \sum_{i < r} a_{ir}(\text{tr}' \rho_1(x_i x_r)) \\
&\quad + \sum_{j < k < p} a_{jpk}(\text{tr}' \rho_1(x_j x_k x_p)) + \sum_{j < p < k} a_{jpk}(\text{tr}' \rho_1(x_j x_p x_k)) \\
&\quad + \sum_{i \neq p < j < k} a_{pj k}(\text{tr}' \rho_1(x_p x_j x_k)) \\
&\quad + \sum_{p < q < i} a_{pqi}(\text{tr}' \rho_1(x_p x_q x_i)) + \sum_{p < i < q} a_{piq}(\text{tr}' \rho_1(x_p x_i x_q)) \\
&\quad + \sum_{i < p < q} a_{ipq}(\text{tr}' \rho_1(x_i x_p x_q)) \\
&= 0.
\end{aligned}$$

By the uniqueness of the power series expansion on D , each of the coefficients of the monomials in $f(\text{tr}' \rho_1(x_{i_1} \cdots x_{i_l}))$ must be equal to zero. Here we observe the coefficients of the monomials of degree less than four.

First, from the coefficient of stu , we obtain $(l^2 - m^2)a_{ijk} = 0$. Since we can choose $l, m \in \mathbb{C}$ arbitrary, we see $a_{ijk} = 0$. Therefore (14) reduces

$$\begin{aligned}
f(\text{tr}' \rho_1(x_{i_1} \cdots x_{i_l})) &= a_i(\text{tr}' \rho_1(x_i)) + a_{jk}(\text{tr}' \rho_1(x_j x_k)) \\
&\quad + \sum_{r < i} a_{ri}(\text{tr}' \rho_1(x_r x_i)) + \sum_{i < r} a_{ir}(\text{tr}' \rho_1(x_i x_r)) \\
&= 0.
\end{aligned}$$

Next, from the coefficient of st , su and tu , we see $a_{ij} = 0$, $a_{ik} = 0$ and $a_{jk} = 0$ respectively. Hence we have

$$f(\text{tr}' \rho_1(x_{i_1} \cdots x_{i_l})) = a_i(\text{tr}' \rho_1(x_i)) = 0.$$

Furthermore, from the coefficient of s^2 , we see $a_i = 0$.

On the other hand, for any $t, u \in D$, define a representation $\rho'_1, \rho''_1 : F_n \rightarrow \text{SL}(2, \mathbb{C})$ by

$$\begin{aligned}
\rho'_1(x_r) &:= \begin{cases} \begin{pmatrix} 1-t & 1 \\ 0 & (1-t)^{-1} \end{pmatrix} & \text{if } r = j, \\ E_2 & \text{if } r \neq j, \end{cases} \\
\rho''_1(x_r) &:= \begin{cases} \begin{pmatrix} 1-u & 1 \\ 0 & (1-u)^{-1} \end{pmatrix} & \text{if } r = k, \\ E_2 & \text{if } r \neq k. \end{cases}
\end{aligned}$$

Then, by an argument similar to that in the above, from the coefficients of t^2 in $f(\text{tr}' \rho'_1(x_{i_1} \cdots x_{i_l}))$, and of u^2 in $f(\text{tr}' \rho''_1(x_{i_1} \cdots x_{i_l}))$, we obtain $a_j = 0$ and $a_k = 0$ respectively.

Therefore we conclude that $f \in J_0^2$. This completes the proof of Proposition 4.11. \square

4.3.2. A basis of $\text{gr}^2(J)$.

Set

$$\begin{aligned} S_1 &:= \{t'_i t'_j \mid 1 \leq i \leq j \leq n\} \cup \{t'_i t'_{ab} \mid 1 \leq i \leq n, 1 \leq a < b \leq n\} \\ &\quad \cup \{t'_i t'_{abc} \mid 1 \leq i \leq n, 1 \leq a < b < c \leq n\} \\ &\quad \cup \{t'_{ij} t'_{ab} \mid 1 \leq i < j \leq n, 1 \leq a < b \leq n, (i, j) \leq (a, b)\}, \\ S_2 &:= \{t'_{ab} t'_{abc}, t'_{ac} t'_{abc}, t'_{bc} t'_{abc} \mid 1 \leq a < b < c \leq n\} \\ &\quad \cup \{t'_{ia} t'_{abc}, t'_{ib} t'_{abc}, t'_{ic} t'_{abc}, t'_{ia} t'_{ibc}, t'_{ab} t'_{iac}, t'_{ab} t'_{ibc}, t'_{ac} t'_{ibc}, t'_{ib} t'_{iac} \mid 1 \leq i < a < b < c \leq n\} \\ &\quad \cup \{t'_{ja} t'_{ibc}, t'_{jb} t'_{iac}, t'_{jc} t'_{iab}, t'_{ab} t'_{ijc}, t'_{ac} t'_{ijb}, t'_{bc} t'_{ija} \mid 1 \leq i < j < a < b < c \leq n\} \end{aligned}$$

and $S := S_1 \cup S_2$. Here we show that $\pi_{\mathbf{Q}}(S)$ forms a basis of $\text{gr}^2(J)$ as a \mathbf{Q} -vector space.

First, we show

Proposition 4.12. $\pi_{\mathbf{Q}}(S)$ generates $\text{gr}^2(J)$.

Proof. Set

$$\begin{aligned} S' &:= \{t'_{ij} t'_{abc} \mid 1 \leq i < j \leq n, 1 \leq a < b < c \leq n\}, \\ S'' &:= \{t'_{ijk} t'_{abc} \mid 1 \leq i < j < k \leq n, 1 \leq a < b < c \leq n, (i, j, k) \leq (a, b, c)\}. \end{aligned}$$

Then $\text{gr}^2(J)$ is generated by $\pi_{\mathbf{Q}}(S_1 \cup S' \cup S'')$. Consider relations

$$(15) \quad (2t_{ijk} - t_i t_{jk} - t_j t_{ik} - t_k t_{ij} + t_i t_j t_k)(2t_{abc} - t_a t_{bc} - t_b t_{ac} - t_c t_{ab} + t_a t_b t_c) \\ = \begin{vmatrix} t_i & t_{ia} & t_{ib} & t_{ic} \\ t_j & t_{ja} & t_{jb} & t_{jc} \\ t_k & t_{ka} & t_{kb} & t_{kc} \\ 2 & t_a & t_b & t_c \end{vmatrix}$$

in $\mathbf{Q}[t]/I_{\mathbf{Q}}$ where $t_{ii} = t_i^2 - 2$. (For details, see Corollary 4.12 in [1].) Substituting $t'_{i_1 \dots i_l} + 2$ to each of $t_{i_1 \dots i_l}$ in the equations above, we verify that $\pi_{\mathbf{Q}}(t'_{ijk} t'_{abc})$ is written as a polynomial of the indeterminates t'_i and t'_{ij} in $\mathbf{Q}[t]/I_{\mathbf{Q}}$. Hence we see that $\pi_{\mathbf{Q}}(S \cup S')$ generates $\text{gr}^2(J)$.

Next, we reduce the generators of $\pi_{\mathbf{Q}}(S')$. Consider a quotient \mathbf{Q} -vector space V of $\text{gr}^2(J)$ by a subspace generated by $\pi_1(S_1)$. We write the equality in V as \doteq . Now, fix $1 \leq i < a < b < c \leq n$. Then we have elements

$$\begin{aligned} p_2 &:= t_i t_{abc} + t_{acb} - t_{abc} - t_{ia} t_{ibc} + t_{ib} t_{iac} - t_{ic} t_{iab} - t_i t_b t_{iac} + t_b t_{ia} t_{ic}, \\ p_3 &:= t_{ib} t_{iabc} - t_{iab} t_{ibc} - t_i t_{iac} + t_{ia} t_{ic} - t_a t_c + 2t_{ac} - t_b t_{abc} + t_{ab} t_{bc}, \\ p_4 &:= t_{iba} t_{iabc} - t_{ia} t_{ab} t_{ibc} - t_{ab} t_{abc} + t_{it} t_{ibc} - t_{ib} t_{ibc} - t_{ia} t_{iac} + t_a t_{ab} t_{bc} + t_a t_{ia} t_{ic} \\ &\quad + t_a t_{ac} - t_i t_{ic} - t_b t_{bc} + 2t_c, \\ (p_3)^{\sigma_{bc}} &:= t_{ic} t_{iabc} - t_{iac} t_{icb} - t_i t_{iab} + t_{ia} t_{ib} - t_a t_b + 2t_{ab} - t_c t_{acb} + t_{ac} t_{cb} \end{aligned}$$

in I due to Whittemore [17], where $\sigma_{bc} \in \text{Aut } F_n$ is an automorphism such that

$$x_r \mapsto \begin{cases} x_c & \text{if } r = b, \\ x_b & \text{if } r = c, \\ x_r & \text{if } r \neq b, c. \end{cases}$$

From the above, by using (12) and a straightforward calculation, we obtain equations

$$(16) \quad \begin{aligned} t'_{ic}t'_{iab} &\doteq -t'_{ia}t'_{ibc} + t'_{ib}t'_{iac}, \\ t'_{bc}t'_{iab} &\doteq -t'_{ab}t'_{ibc} + t'_{ib}t'_{abc} - (t'_{ic}t'_{iab} + t'_{ia}t'_{ibc} - t'_{ib}t'_{iac}) \end{aligned}$$

$$(17) \quad \begin{aligned} &\doteq -t'_{ab}t'_{ibc} + t'_{ib}t'_{abc}, \\ t'_{ac}t'_{iab} &\doteq t'_{ia}t'_{abc} + t'_{ab}t'_{iac} - (t'_{ic}t'_{iab} + t'_{ia}t'_{ibc} - t'_{ib}t'_{iac}) \\ &\quad + (-t'_{bc}t'_{iab} - t'_{ab}t'_{ibc} + t'_{ib}t'_{abc}) \end{aligned}$$

$$(18) \quad \begin{aligned} &\doteq t'_{ia}t'_{abc} + t'_{ab}t'_{iac}, \\ t'_{bc}t'_{iac} &\doteq -t'_{ic}t'_{abc} + t'_{ac}t'_{ibc} + (t'_{ic}t'_{iab} + t'_{ia}t'_{ibc} - t'_{ib}t'_{iac}) \end{aligned}$$

$$(19) \quad \doteq -t'_{ic}t'_{abc} + t'_{ac}t'_{ibc}$$

in V respectively. Hence we can remove $t'_{ic}t'_{iab}$, $t'_{bc}t'_{iab}$, $t'_{ac}t'_{iab}$ and $t'_{bc}t'_{iac}$ from the generating set $\pi_{\mathbf{Q}}(S')$.

Fix $1 \leq i < j < a < b < c \leq n$. Using (6), we have

$$\begin{aligned} 2t_{(ij)abc} &= t_{ij}t_{abc} + t_at_{ijbc} + t_bt_{ijac} + t_ct_{ijab} + t_{ija}t_{bc} - t_{ijb}t_{ac} + t_{ijc}t_{ab} \\ &\quad - t_{ijt_a}t_{bc} - t_at_bt_{ijc} - t_{ij}t_ct_{ab} - t_bt_ct_{ija} + t_{ij}t_at_bt_c \end{aligned}$$

in $\mathbf{Z}[t]$. On the other hand, we have

$$\begin{aligned} 2t_{(ja)bci} &= t_{ja}t_{bci} + t_bt_{jaci} + t_ct_{jab} + t_it_{jab} + t_{jab}t_{ci} - t_{jac}t_{bi} + t_{jai}t_{bc} \\ &\quad - t_{ja}t_bt_{ci} - t_bt_ct_{jai} - t_{ja}t_it_{bc} - t_ct_it_{jab} + t_{ja}t_bt_ct_i. \end{aligned}$$

Hence, from $2t_{(ij)abc} = 2t_{(ja)bci}$, by using (6) again, we obtain

$$(20) \quad t'_{ij}t'_{abc} + t'_{ib}t'_{jac} - t'_{ic}t'_{jab} \doteq t'_{ja}t'_{ibc} - t'_{ab}t'_{ijc} + t'_{ac}t'_{ijb}.$$

Similarly, from equations $2t_{(ij)abc} = 2t_{(ab)cij}$, $2t_{(ij)abc} = 2t_{(bc)ija}$ and $2t_{(ij)abc} = 2t_{(ci)jab}$, we obtain

$$(21) \quad t'_{ic}t'_{jab} \doteq t'_{jc}t'_{iab} - t'_{ac}t'_{ijb} + t'_{bc}t'_{ija},$$

$$(22) \quad t'_{ia}t'_{jbc} \doteq t'_{ja}t'_{ibc} - t'_{ab}t'_{ijc} + t'_{ac}t'_{ijb},$$

$$(23) \quad t'_{ij}t'_{abc} - t'_{ic}t'_{jab} \doteq t'_{ja}t'_{ibc} - t'_{jb}t'_{iac} + t'_{ac}t'_{ijb} - t'_{bc}t'_{ija}$$

respectively. From (20), (21) and (23), we see

$$(24) \quad t'_{ib}t'_{jac} \doteq -t'_{ab}t'_{ijc} + t'_{jb}t'_{iac} + t'_{bc}t'_{ija}$$

$$(25) \quad t'_{ij}t'_{abc} \doteq t'_{ja}t'_{ibc} - t'_{jb}t'_{iac} + t'_{jc}t'_{iab}.$$

Therefore, by observing (25), (22), (24) and (21), we can remove $t'_{ij}t'_{abc}$, $t'_{ia}t'_{jbc}$, $t'_{ib}t'_{jac}$ and $t'_{ic}t'_{jab}$ from the generating set $\pi_{\mathbf{Q}}(S')$.

Then we obtain the required result. This completes the proof of Proposition 4.12. \square

Next we prove

Proposition 4.13. *Elements in $\pi_{\mathbf{Q}}(S)$ are linearly independent in $\text{gr}^2(J)$.*

Proof. Set

$$\begin{aligned}
g := & \sum_{1 \leq i \leq j \leq n} d_{i,j} t'_i t'_j + \sum_{\substack{1 \leq i \leq n, \\ 1 \leq a < b \leq n}} d_{i,ab} t'_i t'_{ab} + \sum_{\substack{1 \leq i \leq n, \\ 1 \leq a < b < c \leq n}} d_{i,abc} t'_i t'_{abc} \\
& + \sum_{(1,2) \leq (i,j) \leq (a,b) \leq (n-1,n)} d_{ij,ab} t'_{ij} t'_{ab} + \sum_{\substack{1 \leq i < j \leq n, 1 \leq a < b < c \leq n; \\ t'_{ij} t'_{abc} \in S_2}} d_{ij,abc} t'_{ij} t'_{abc} \in \mathbf{Q}[t]
\end{aligned}$$

for $d_{i,j}, d_{i,ab}, d_{i,abc}, d_{ij,ab}, d_{ij,abc} \in \mathbf{Q}$. Assume $\pi_{\mathbf{Q}}(g) \in J^3$. Then, it suffices to show that $d_{i,j} = d_{i,ab} = d_{i,abc} = d_{ij,ab} = d_{ij,abc} = 0$.

Step 1. $d_{ij,abc} = 0$ for any $1 \leq i < j \leq n$ and $1 \leq a < b < c \leq n$ such that $t'_{ij} t'_{abc} \in S_2$.

Set $N_1 := \{i, j, a, b, c\}$. We consider three cases according to the number of elements in N_1 .

Case 1-1. $\#N_1 = 3$.

Assume $N_1 = \{a, b, c\}$ and $a < b < c$. We show

$$(26) \quad d_{ab,abc} = d_{ac,abc} = d_{bc,abc} = 0.$$

To do this, for any $k, l, m, s, t, u \in \mathbf{C}$, consider a representation $\rho_2 : F_n \rightarrow \mathrm{SL}(2, \mathbf{C})$ defined by

$$\rho_2(x_a) := \begin{pmatrix} 1 - ks & k^2 s \\ -s & 1 + ks \end{pmatrix}, \quad \rho_2(x_b) := \begin{pmatrix} 1 - lt & l^2 t \\ -t & 1 + lt \end{pmatrix}, \quad \rho_2(x_c) := \begin{pmatrix} 1 - mu & m^2 u \\ -u & 1 + mu \end{pmatrix}$$

and $\rho_2(x_r) = E_2$ for $r \neq a, b, c$. Then we have

$$\mathrm{tr}' \rho_2(x_a x_b) = -(k - l)^2 st,$$

$$\mathrm{tr}' \rho_2(x_a x_b x_c) = (k - l)(l - m)(m - k)stu - (k - l)^2 st - (l - m)^2 tu - (m - k)^2 su,$$

and so on.

Consider each of $\mathrm{tr}' \rho_2(x_{i_1} \cdots x_{i_l})$ as a polynomial of s, t, u with rational coefficients. Then by the observation above, each of them is zero, or is of degree greater than one. Hence, since $\pi_{\mathbf{Q}}(g) \in J^3$, the degree of $g(\mathrm{tr}' \rho_2(x_{i_1} \cdots x_{i_l}))$ must be greater than five. This shows that each of the coefficients of $s^2 t^2 u, s^2 t u^2, s t^2 u^2$ in $g(\mathrm{tr}' \rho_2(x_{i_1} \cdots x_{i_l}))$ is equal to zero. Hence we see

$$\begin{aligned}
& -(k - l)^2 (k - l)(l - m)(m - k) d_{ab,abc} - (k - m)^2 (k - l)(l - m)(m - k) d_{ac,abc} \\
& - (l - m)^2 (k - l)(l - m)(m - k) d_{bc,abc} \\
& = 0.
\end{aligned}$$

Furthermore, since we can choose $k, l, m \in \mathbf{C}$ arbitrary, we obtain (26).

Case 1-2. $\#N_1 = 4$.

Assume $N_1 = \{i, a, b, c\}$ and $i < a < b < c$. It suffices to show

$$\begin{aligned}
(27) \quad & d_{ia,abc} = d_{ab,iac} = 0, \quad d_{ib,abc} = d_{ab,ibc} = 0, \\
& d_{ic,abc} = d_{ac,ibc} = 0, \quad d_{ia,ibc} = d_{ib,iac} = 0.
\end{aligned}$$

To do this, for any $k, l, m, p, s, t, u, v \in \mathbf{C}$, consider a representation $\rho_3 : F_n \rightarrow \text{SL}(2, \mathbf{C})$ defined by

$$\rho_3(x_i) := \begin{pmatrix} 1 - pv & p^2v \\ -v & 1 + pv \end{pmatrix}$$

and $\rho_3(x_r) := \rho_2(x_r)$ for $r \neq i$. From the coefficient of s^2tuv in $g(\text{tr}' \rho_3(x_{i_1} \cdots x_{i_l}))$, we have

$$-(k-p)^2(k-l)(l-m)(m-k)d_{ia,abc} - (k-l)^2(p-k)(k-m)(m-p)d_{ab,iac} = 0$$

and from the coefficients of k^4l, k^4p , we see

$$d_{ia,abc} = d_{ab,iac} = 0.$$

Similarly, by observing the coefficients of st^2uv, stu^2v and $stuv^2$ in $g(\text{tr}' \rho_3(x_{i_1} \cdots x_{i_l}))$, we see

$$\begin{aligned} -(l-p)^2(k-l)(l-m)(m-k)d_{ib,abc} - (l-k)^2(p-l)(l-m)(m-p)d_{ab,ibc} &= 0, \\ -(p-m)^2(k-l)(l-m)(m-k)d_{ic,abc} - (k-m)^2(p-l)(l-m)(m-p)d_{ac,ibc} &= 0, \\ -(p-k)^2(p-l)(l-m)(m-p)d_{ia,ibc} - (p-l)^2(p-k)(k-m)(m-p)d_{ib,iac} &= 0 \end{aligned}$$

respectively. From these, we obtain (27).

Case 1-3. $\sharp N_1 = 5$.

Assume $i < j < a < b < c$. Then it suffices to show

$$(28) \quad d_{ja,ibc} = d_{jb,iac} = d_{jc,iab} = d_{ab,ijc} = d_{ac,ijb} = d_{bc,ija} = 0.$$

To begin with, for any $k, l, m, p, q, s, t, u, v, w \in \mathbf{C}$, consider a representation $\rho_4 : F_n \rightarrow \text{SL}(2, \mathbf{C})$ defined by

$$\rho_4(x_j) := \begin{pmatrix} 1 - qw & q^2w \\ -w & 1 + qw \end{pmatrix}$$

and $\rho_4(x_r) := \rho_3(x_r)$ for $r \neq j$.

Then $g(\text{tr}' \rho_4(x_{i_1} \cdots x_{i_l})) \in \mathbf{C}$ is written as a polynomial of $k, l, m, p, q, s, t, u, v, w$ with rational coefficients. By observing the coefficient of $stuvw$ in $g(\text{tr}' \rho_4(x_{i_1} \cdots x_{i_l}))$, we have

$$\begin{aligned} &-(k-q)^2(p-l)(l-m)(m-p)d_{ja,ibc} - (l-q)^2(p-k)(k-m)(m-p)d_{jb,iac} \\ &-(q-m)^2(p-k)(k-l)(l-p)d_{jc,iab} - (k-l)^2(p-q)(q-m)(m-p)d_{ab,ijc} \\ &-(k-m)^2(p-q)(q-l)(l-p)d_{ac,ijb} - (l-m)^2(p-q)(q-k)(k-p)d_{bc,ija} \\ &= 0. \end{aligned}$$

Furthermore, from the coefficients of $p^2m^2k, p^2m^2l, p^2l^2m, p^2q^2l, p^2q^2m, p^2l^2q, p^2k^2m$, we obtain

$$\begin{aligned} d_{bc,ija} &= d_{jc,iab}, & d_{ac,ijb} &= -d_{jc,iab}, & d_{ab,ijc} &= -d_{jb,iac}, \\ d_{jc,iab} &= d_{ja,ibc}, & d_{jb,iac} &= -d_{ja,ibc}, & d_{ab,ijc} &= -d_{bc,ija}, & d_{ja,ibc} &= -d_{ab,ijc} \end{aligned}$$

respectively. From this, we obtain (28).

Step 2. $d_{ij,ab} = 0$ for any $1 \leq i < j \leq n$ and $1 \leq a < b \leq n$ and $(i, j) \leq (a, b)$.

Set $N_2 := \{i, j, a, b\}$. We consider three cases according to the number of elements in N_2 .

First, we consider the case where $\sharp N_2 = 2$. We show $d_{ab,ab} = 0$ for any $1 \leq a < b \leq n$. Recall the representation ρ_2 . By observing the coefficients of $s^2 t^2$ in $g(\text{tr}' \rho_2(x_{i_1} \cdots x_{i_l}))$, we obtain

$$(k-l)^4 d_{ab,ab} = 0.$$

This shows that $d_{ab,ab} = 0$.

Next, consider the case where $\sharp N_2 = 3$. It suffices to show that for any $1 \leq i < a < b \leq n$,

$$(29) \quad d_{ia,ab} = d_{ib,ab} = d_{ia,ib} = 0.$$

By observing the coefficients of $s^2 tv$, $st^2 v$ and stv^2 in $g(\text{tr}' \rho_3(x_{i_1} \cdots x_{i_l}))$, we obtain

$$\begin{aligned} (p-k)^2(k-l)^2 d_{ia,ab} &= 0, & (p-l)^2(k-l)^2 d_{ib,ab} &= 0, \\ (p-k)^2(p-l)^2 d_{ia,ib} &= 0. \end{aligned}$$

From the coefficients of k^4 , l^4 and p^4 , we see (29).

Finally, consider the case where $\sharp N_2 = 4$. It suffices to show that for any $1 \leq i < j < a < b \leq n$,

$$(30) \quad d_{ij,ab} = d_{ia,jb} = d_{ib,ja} = 0.$$

By observing the coefficients of $stuv$ in $g(\text{tr}' \rho_4(x_{i_1} \cdots x_{i_l}))$, we obtain

$$(p-q)^2(k-l)^2 d_{ij,ab} + (p-k)^2(q-l)^2 d_{ia,jb} = 0 + (p-l)^2(q-k)^2 d_{ib,ja} = 0.$$

From the coefficients of pqk^2 , qlk^2 and plq^2 , we see (30).

Step 3. $d_{i,abc} = 0$ for any $1 \leq i \leq n$ and $1 \leq a < b < c \leq n$.

First, assume $i \neq a, b, c$. For any $v \in D$ and $k, l, m, s, t, u \in \mathbf{C}$, consider a representation $\rho_5 : F_n \rightarrow \text{SL}(2, \mathbf{C})$ defined by

$$\rho_5(x_i) := \begin{pmatrix} 1-v & 0 \\ 0 & (1-v)^{-1} \end{pmatrix}$$

and $\rho_5(x_r) := \rho_3(x_r)$ for $r \neq i$. By observing the coefficients of $stuv^2$ in $g(\text{tr}' \rho_5(x_{i_1} \cdots x_{i_l}))$, we obtain

$$(k-l)(l-m)(m-k) d_{i,abc} = 0.$$

This shows that $d_{i,abc} = 0$.

Next, we consider the case where $i = a, b$ or c . For any $s \in D$ and $l, m, t, u \in \mathbf{C}$, consider a representation $\rho_6 : F_n \rightarrow \text{SL}(2, \mathbf{C})$ defined by

$$\rho_6(x_a) := \begin{pmatrix} 1-s & 0 \\ 0 & (1-s)^{-1} \end{pmatrix}$$

and $\rho_6(x_r) := \rho_2(x_r)$ for $r \neq a$. Then from the coefficients of $s^3 tu$ in $g(\text{tr}' \rho_6(x_{i_1} \cdots x_{i_l}))$, we have

$$(l^2 - m^2) d_{a,abc} - (l-m)^2 d_{a,bc} = 0.$$

This shows $d_{a,abc} = d_{a,bc} = 0$. Similarly, we can obtain

$$d_{b,abc} = d_{c,abc} = 0.$$

Step 4. $d_{i,ab} = 0$ for any $1 \leq i \leq n$ and $1 \leq a < b \leq n$.

Assume $i \neq a, b$. By observing the coefficients of stv^2 in $g(\text{tr}' \rho_5(x_{i_1} \cdots x_{i_l}))$, we obtain

$$-(k-l)^2 d_{i,ab} = 0.$$

This shows $d_{i,ab} = 0$. (Remark that for $i < a < b$, this has already been obtained in Step 3.)

On the other hand, from the coefficients of s^3t in $g(\text{tr}' \rho_6(x_{i_1} \cdots x_{i_l}))$, we have

$$2ld_{a,ab} = 0.$$

This shows $d_{a,ab} = 0$. Similarly, we can obtain

$$d_{b,ab} = 0.$$

Step 5. $d_{i,a} = 0$ for any $1 \leq i, a \leq n$.

Assume $i \neq a$. For any $s, v \in D$, consider a representation $\rho_7 : F_n \rightarrow \text{SL}(2, \mathbf{C})$ defined by

$$\rho_7(x_i) := \begin{pmatrix} 1-s & 0 \\ 0 & (1-s)^{-1} \end{pmatrix}, \quad \rho_7(x_a) := \begin{pmatrix} 1-v & 0 \\ 0 & (1-v)^{-1} \end{pmatrix}$$

and $\rho_7(x_r) := E_2$ for $r \neq i, a$. Then by observing the coefficients of s^4 , s^2v^2 and v^4 in $g(\text{tr}' \rho_7(x_{i_1} \cdots x_{i_l}))$, we obtain

$$d_{i,i} = 0, \quad d_{i,a} = 0, \quad \text{and} \quad d_{a,a} = 0$$

respectively.

Therefore we have obtained all coefficients of g are equal to zero. This completes the proof of Proposition 4.13. \square

5. A CENTRAL FILTRATION $\mathcal{E}_G(k)$

In this section, for any group G , we introduce a descending filtration of $\text{Aut } G$ consisting of its normal subgroups. This is an analogue of the Andreadakis-Johnson filtration of $\text{Aut } G$. (For details for the Andreadakis-Johnson filtration, see [14] or [15], for example.)

5.1. Definition of $\mathcal{E}_G(k)$.

For any $k \geq 1$, let

$$\mathcal{E}_G(k) := \text{Ker}(\text{Aut } G \rightarrow \text{Aut}(J/J^{k+1}))$$

be the kernel of a homomorphism $\text{Aut } G \rightarrow \text{Aut}(J/J^{k+1})$ which is induced from the action of $\text{Aut } G$ on J/J^{k+1} . Then the groups $\mathcal{E}_G(k)$ define a descending filtration

$$\mathcal{E}_G(1) \supset \mathcal{E}_G(2) \supset \cdots \supset \mathcal{E}_G(k) \supset \cdots$$

of $\text{Aut } G$. Here we show that this is a central filtration.

For any $f \in J$ and $\sigma \in \text{Aut } G$, set

$$s_\sigma(f) := f^\sigma - f \in J.$$

Then we have

Lemma 5.1. *For any $f \in J$ and $\sigma, \tau \in \text{Aut } G$,*

- (1) $s_{\sigma\tau}(f) = (s_\sigma(f))^\tau + s_\tau(f)$,
- (2) $s_{1_G}(f) = 0$,
- (3) $s_{\sigma^{-1}}(f) = -(s_\sigma(f))^{\sigma^{-1}}$,
- (4) $s_{[\sigma, \tau]}(f) = \{s_\tau(s_\sigma(f)) - s_\sigma(s_\tau(f))\}^{\sigma^{-1}\tau^{-1}}$.

Proof. The part of (1), (2) and (3) is straightforward. Here we prove the part (4). Using (1), (2) and (3), we obtain

$$\begin{aligned}
s_{[\sigma, \tau]}(f) &\stackrel{(1)}{=} (s_{\sigma\tau}(f))^{\sigma^{-1}\tau^{-1}} + s_{\sigma^{-1}\tau^{-1}}(f) \\
&\stackrel{(3)}{=} (s_{\sigma\tau}(f))^{\sigma^{-1}\tau^{-1}} - (s_{\tau\sigma}(f))^{\sigma^{-1}\tau^{-1}} \\
&\stackrel{(1)}{=} \{(s_\sigma(f))^\tau + s_\tau(f) - (s_\tau(f))^\sigma - s_\sigma(f)\}^{\sigma^{-1}\tau^{-1}} \\
&= \{s_\tau(s_\sigma(f)) - s_\sigma(s_\tau(f))\}^{\sigma^{-1}\tau^{-1}}.
\end{aligned}$$

This completes the proof of Lemma 5.1. \square

Lemma 5.2. *For any $k, l \geq 1$, $f \in J^l$ and $\sigma \in \mathcal{E}_G(k)$, we have $s_\sigma(f) \in J^{k+l}$.*

Proof. It suffices to show the lemma for the case where f is (the coset class of) a monomial $t'_{a_1 \dots a_{r_1}} t'_{b_1 \dots b_{r_2}} \dots t'_{c_1 \dots c_{r_l}}$. Then we have

$$\begin{aligned}
s_\sigma(f) &= f^\sigma - f \\
&= (t'_{a_1 \dots a_{r_1}})^\sigma \dots (t'_{c_1 \dots c_{r_l}})^\sigma - t'_{a_1 \dots a_{r_1}} \dots t'_{c_1 \dots c_{r_l}} \\
&= (t'_{a_1 \dots a_{r_1}} + s_\sigma(t'_{a_1 \dots a_{r_1}})) \dots (t'_{c_1 \dots c_{r_l}} + s_\sigma(t'_{c_1 \dots c_{r_l}})) - t'_{a_1 \dots a_{r_1}} \dots t'_{c_1 \dots c_{r_l}}.
\end{aligned}$$

By the definition of $\mathcal{E}_G(k)$, the elements $s_\sigma(t'_{a_1 \dots a_{r_1}}), \dots, s_\sigma(t'_{c_1 \dots c_{r_l}})$ belong to J^{k+1} . Therefore, we obtain $s_\sigma(f) \in J^{k+l}$. This completes the proof of Lemma 5.2. \square

Proposition 5.3. *For any $k, l \geq 1$, $[\mathcal{E}_G(k), \mathcal{E}_G(l)] \subset \mathcal{E}_G(k+l)$.*

Proof. For any $\sigma \in \mathcal{E}_G(k)$, $\tau \in \mathcal{E}_G(l)$ and $f \in J$, by Lemmas 5.1 and 5.2, we see

$$\begin{aligned}
s_{[\sigma, \tau]}(f) &= \{s_\tau(s_\sigma(f)) - s_\sigma(s_\tau(f))\}^{\sigma^{-1}\tau^{-1}} \\
&\equiv 0 \pmod{J^{k+l+1}}.
\end{aligned}$$

Hence $[\sigma, \tau] \in \mathcal{E}_G(k+l)$. This completes the proof of Proposition 5.3. \square

This proposition shows that the filtration $\mathcal{E}_G(k)$ is a central filtration of $\text{Aut } G$. Next, our interests is how different the filtration $\mathcal{E}_G(k)$ is from the Andreadakis-Johnson filtration $\mathcal{A}_G(k)$. We consider this problem for the case where $G = F_n$.

5.2. The group $\mathcal{E}_{F_n}(1)$.

Here we show that $\mathcal{E}_{F_n}(1) = \text{Inn } F_n \cdot \mathcal{A}_{F_n}(2)$. First, we show that $\mathcal{E}_{F_n}(1)$ is contained in the IA-automorphism group $\text{IA}_n = \mathcal{A}_{F_n}(1)$. In the following, we always identify $t'_{i_1 \dots i_l} \in \mathbf{Q}[t]/I_{\mathbf{Q}}$ with $\text{tr}' x_{i_1} \dots x_{i_l} \in \mathcal{F}(R(G), \mathbf{C})$ through $\pi_{\mathbf{Q}}$.

To begin with, we prepare some lemmas.

Lemma 5.4. For any $s \in \mathbf{C}$, Set $A := \begin{pmatrix} s+2 & 1 \\ -1 & 0 \end{pmatrix}$, and

$$\mathrm{tr} A^m = x_m^{(0)} + x_m^{(1)}s + x_m^{(2)}s^2 + \dots$$

for any $m \in \mathbf{Z}$. Then $x_m^{(0)} = 2$ and $x_m^{(1)} = m^2$.

This lemma is obtained by a straightforward calculation.

Lemma 5.5. For any $1 \leq i \leq n$ and a word $w := x_{i_1}^{e_1} \dots x_{i_l}^{e_l} \in F_n$, assume

$$\mathrm{tr}'(w) \equiv \mathrm{tr}' x_i \pmod{J^2}.$$

Then we have

$$\left(\sum_{i_j=i} e_j\right)^2 = 1, \quad \sum_{i_j=k} e_j = 0$$

for any $k \neq i$.

Proof. For any $s \in \mathbf{C}$, consider a representation $\rho_8 : F_n \rightarrow \mathrm{SL}(2, \mathbf{C})$ defined by

$$\rho_8(x_r) = \begin{cases} A, & \text{if } r = i, \\ E_2, & \text{if } r \neq i. \end{cases}$$

Then from $\mathrm{tr}'(w) \equiv \mathrm{tr}' x_i \pmod{J^2}$ and Lemma 5.5, we obtain

$$m^2 s + (\text{terms of degree } \geq 2) = s + (\text{terms of degree } \geq 2)$$

where $m = \sum_{i_j=i} e_j$. Hence $m^2 = 1$.

Similarly, for any $k \neq i$ and $s \in \mathbf{C}$, considering a representation $\rho_9 : F_n \rightarrow \mathrm{SL}(2, \mathbf{C})$ defined by

$$\rho_9(x_r) = \begin{cases} A, & \text{if } r = k, \\ E_2, & \text{if } r \neq k. \end{cases}$$

we obtain $\sum_{i_j=k} e_j = 0$. This completes the proof of Lemma 5.5. \square

From this lemma, we see that for any $\sigma \in \mathcal{E}_{F_n}(1)$ and $1 \leq i \leq n$,

$$x_i^\sigma = x_i^{m_i} c_i, \quad m_i = \pm 1$$

for some $c_i \in \Gamma_{F_n}(2)$. Next, we show

Lemma 5.6. For any $\sigma \in \mathcal{E}_{F_n}(1)$, $m_1 = m_2 = \dots = m_n$.

Proof. Choose any $1 \leq i < j \leq n$. Consider a representation $\rho_{10} : F_n \rightarrow \mathrm{SL}(2, \mathbf{C})$ defined by

$$\rho_{10}(x_r) = \begin{cases} A, & \text{if } r = i, j, \\ E_2, & \text{if } r \neq i, j. \end{cases}$$

Then from $\mathrm{tr}'((x_i x_j)^\sigma) \equiv \mathrm{tr}' x_i x_j \pmod{J^2}$, we see

$$(m_i + m_j)^2 s + (\text{terms of degree } \geq 2) = 4s + (\text{terms of degree } \geq 2).$$

Hence we obtain $m_i = m_j$. This completes the proof of Lemma 5.6. \square

Therefore we see that for any $\sigma \in \mathcal{E}_{F_n}(1)$ and $1 \leq i \leq n$,

$$x_i^\sigma = x_i^{m_\sigma} c_i, \quad m_\sigma = \pm 1$$

for some $c_i \in \Gamma_{F_n}(2)$. Here assume

$$c_i \equiv [x_2, x_1]^{e_{21}(i)} [x_3, x_1]^{e_{31}(i)} \cdots [x_n, x_{n-1}]^{e_{n \ n-1}(i)} \\ \cdot [x_2, x_1, x_1]^{e_{211}(i)} \cdots [x_n, x_{n-1}, x_n]^{e_{n \ n-1 \ n}(i)} \pmod{\Gamma_{F_n}(4)}$$

for $e_{ba}(i), e_{bac}(i) \in \mathbf{Z}$. Here in the right hand side of the equation, terms $[x_b, x_a]$ for $b > a$ are multiplied according to the lexicographic ordering

$$(b, a) < (b', a') \iff a < a' \text{ or, } a = a' \text{ and } b < b',$$

and terms $[x_b, x_a, x_c]$ for $b > a \leq c$ are multiplied according to the lexicographic ordering

$$(b, a, c) < (b', a', c') \iff \begin{cases} a < a', \\ a = a' \text{ and } b < b' \text{ or,} \\ a = a', b = b' \text{ and } c < c'. \end{cases}$$

We remark that

$$\{[x_b, x_a] \mid 1 \leq a < b \leq n\} \text{ and } \{[x_b, x_a, x_c] \mid b > a \leq c\}$$

form basis of $\Gamma_{F_n}(2)/\Gamma_{F_n}(3)$ and $\Gamma_{F_n}(3)/\Gamma_{F_n}(4)$ as free abelian groups respectively. (For details, see [4], for example.)

Lemma 5.7. *As the notation above, $e_{ba}(i) = 0$ if $a, b \neq i$.*

Proof. We prove this lemma for the case where $a < b < i$. The other cases are proved in a similar way. For any $s, t, u \in \mathbf{C}$, consider a representation $\rho_{11} : F_n \rightarrow \text{SL}(2, \mathbf{C})$ defined by

$$\rho_{11}(x_a) := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \rho_{11}(x_b) := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \rho_{11}(x_i) := \begin{pmatrix} 1-u & u \\ -u & 1+u \end{pmatrix}$$

and $\rho_{11}(x_r) = E_2$ for $r \neq a, b, i$. Then for any $e \in \mathbf{Z}$, we have

$$\begin{aligned} \rho_{11}([x_b, x_a]^e) &\equiv \begin{pmatrix} 1-est & es^2t \\ -est^2 & 1+est \end{pmatrix}, \\ \rho_{11}([x_i, x_a]^e) &\equiv \begin{pmatrix} 1+e(su-su^2) & -e(2su+s^2u-su^2) \\ -esu^2 & 1-e(su-su^2) \end{pmatrix}, \\ \rho_{11}([x_i, x_b]^e) &\equiv \begin{pmatrix} 1+e(tu+tu^2) & -etu^2 \\ e(2tu+tu^2+t^2u) & 1-e(tu+tu^2) \end{pmatrix}, \\ \rho_{11}([x_b, x_a, x_a]^e) &\equiv \begin{pmatrix} 1 & -2es^2t \\ 0 & 1 \end{pmatrix}, & \rho_{11}([x_b, x_a, x_b]^e) &\equiv \begin{pmatrix} 1 & 0 \\ 2est^2 & 1 \end{pmatrix}, \\ \rho_{11}([x_b, x_a, x_i]^e) &\equiv \begin{pmatrix} 1 & -e(s^2t+2stu) \\ -2estu & 1 \end{pmatrix}, & \rho_{11}([x_i, x_a, x_a]^e) &\equiv \begin{pmatrix} 1 & 2es^2u \\ 0 & 1 \end{pmatrix}, \\ \rho_{11}([x_i, x_a, x_b]^e) &\equiv \begin{pmatrix} 1-2estu & e(s^2u-su^2) \\ -2estu & 1+2estu \end{pmatrix}, \\ \rho_{11}([x_i, x_a, x_i]^e) &\equiv \begin{pmatrix} 1+2esu^2 & -2esu^2 \\ 2esu^2 & 1-2esu^2 \end{pmatrix}, & \rho_{11}([x_i, x_b, x_b]^e) &\equiv \begin{pmatrix} 1 & 0 \\ -2et^2u & 1 \end{pmatrix}, \\ \rho_{11}([x_b, x_a, x_b]^e) &\equiv \begin{pmatrix} 1-etu^2 & 2etu^2 \\ 0 & 1+etu^2 \end{pmatrix} \end{aligned}$$

modulo

$$\{F = (f_{ij}) \mid \text{each of } f_{ij} \text{ is a polynomial of } s, t, u \text{ of degree } \geq 4\}.$$

Hence from $\text{tr}'(x_i)^\sigma \equiv \text{tr}'x_i \pmod{J^2}$, we have

$$\begin{aligned} & \text{tr}'\left(\begin{pmatrix} 1 - m_\sigma u & m_\sigma u \\ -m_\sigma u & 1 + m_\sigma u \end{pmatrix} \rho_{11}([x_2, x_1]^{e_{21}(i)} [x_3, x_1]^{e_{31}(i)} \cdots [x_n, x_{n-1}, x_n]^{e_{n \ n-1 \ n}(i)})\right) \\ &= 0 + (\text{terms of degree } \geq 4). \end{aligned}$$

Therefore, by observing the coefficient of stu , we obtain $2m_\sigma e_{ba}(i) = 0$, and hence $e_{ba}(i) = 0$. This completes the proof of Lemma 5.7. \square

From this lemma, we see that for any $\sigma \in \mathcal{E}_{F_n}(1)$ and $1 \leq i \leq n$,

$$\begin{aligned} c_i &\equiv [x_i, x_1]^{e_{i1}(i)} [x_i, x_2]^{e_{i2}(i)} \cdots [x_i, x_{i-1}]^{e_{ii-1}(i)} \\ &\quad \cdot [x_{i+1}, x_i]^{e_{i+1 \ i}(i)} \cdots [x_n, x_i]^{e_{ni}(i)} \pmod{\Gamma_{F_n}(3)}. \end{aligned}$$

Next, we show

Lemma 5.8. *As the notation above, for any $1 \leq i \leq n$,*

$$e_{i1}(1) = e_{i2}(2) = \cdots = e_{ii-1}(i-1) = -e_{i+1 \ i}(i+1) = \cdots = -e_{ni}(n).$$

Proof. Choose any $a < b < i$, and fix it. We have

$$\begin{aligned} (x_a x_b)^\sigma &= x_a^{m_\sigma} c_a x_b^{m_\sigma} c_b \equiv x_a^{m_\sigma} x_b^{m_\sigma} c_a c_b \pmod{\Gamma_{F_n}(3)}, \\ &\equiv x_a^{m_\sigma} x_b^{m_\sigma} [x_2, x_1]^{e_{21}(a)+e_{21}(b)} \cdots [x_n, x_{n-1}]^{e_{n \ n-1}(a)+e_{n \ n-1}(b)} \pmod{\Gamma_{F_n}(3)}. \end{aligned}$$

By the same argument as that in Lemma 5.7, from an equation $\text{tr}'\rho_{11}((x_a x_b)^\sigma) \equiv \text{tr}'\rho_{11}(x_a x_b) \pmod{J^2}$, we see

$$st + 2m_\sigma(-e_{ia}(a) - e_{ia}(b) + e_{ib}(a) + e_{ib}(b))stu \equiv st + (\text{terms of degree } \geq 4).$$

Hence we obtain $e_{ib}(b) = e_{ia}(a)$.

Next, choose any $a < i < b$, and fix it. Similarly, from an equation $\text{tr}'\rho_{11}((x_a x_b)^\sigma) \equiv \text{tr}'\rho_{11}(x_a x_b) \pmod{J^2}$, we see

$$st + 2m_\sigma(-e_{ia}(a) - e_{ia}(b) - e_{bi}(a) - e_{bi}(b))stu \equiv st + (\text{terms of degree } \geq 4).$$

Hence we obtain $e_{bi}(b) = -e_{ia}(a)$. This completes the proof of Lemma 5.8. \square

For any $1 \leq i \leq n$, set $e_i := e_{i1}(1) = \cdots = -e_{ni}(n)$. By Lemma 5.8, we see

$$x_i^\sigma \equiv x_i^{m_\sigma} [x_1, x_i]^{e_1} \cdots [x_{i-1}, x_i]^{e_{i-1}} [x_{i+1}, x_i]^{e_{i+1}} \cdots [x_n, x_i]^{e_n} \pmod{\Gamma_{F_n}(3)}.$$

Then we show

Lemma 5.9. *As the notation above, for any $\sigma \in \mathcal{E}_{F_n}(1)$, $m_\sigma = 1$.*

Proof. Assume $m_\sigma = -1$. For any $1 \leq j \leq n$, let $\iota_j \in \text{Inn } F_n$ be an inner automorphism of F_n defined by $x \mapsto x_j x x_j^{-1}$ for any $x \in F_n$. Then for any $1 \leq i \leq n$ and $e \in \mathbf{Z}$, we have $x_i^{\iota_j^e} \equiv [x_j, x_i]^e x_i \pmod{\Gamma_{F_n}(3)}$. An element $\sigma' := \sigma \iota_1^{e_1} \cdots \iota_n^{e_n} \in \mathcal{E}_{F_n}(1)$ satisfies

$$x_i^{\sigma'} \equiv x_i^{-1} \pmod{\Gamma_{F_n}(3)}$$

for each $1 \leq i \leq n$. Hence, for any $a < b < i$, we see $(x_a x_b x_i)^{\sigma'} \equiv x_a^{-1} x_b^{-1} x_i^{-1} \pmod{\Gamma_{F_n}(3)}$.

By an argument similar to that in Lemma 5.8, from an equation $\text{tr}'\rho_{11}((x_ax_bx_i)^{\sigma'}) \equiv \text{tr}'\rho_{11}(x_ax_bx_i) \pmod{J^2}$, we obtain

$$st + tu - su + stu = st + tu - su - stu + (\text{terms of degree } \geq 4),$$

and hence the contradiction. This completes the proof of Lemma 5.9. \square

As a corollary, we see

Corollary 5.10. *For any $n \geq 3$, $\mathcal{E}_{F_n}(1) \subset \text{IA}_n$.*

Now, we have

Lemma 5.11. *For any $n \geq 3$, $\mathcal{A}_{F_n}(2) \subset \mathcal{E}_{F_n}(1)$*

Proof. For any $\sigma \in \mathcal{A}_{F_n}(2)$, and any $x \in F_n$, we have $x^\sigma = xy$ for some $y \in \Gamma_{F_n}(3)$. Since $\Gamma_{F_n}(3)$ is generated by elements type of $[a, b, c]$ for $a, b, c \in F_n$, we can write

$$y = [a_1, b_1, c_1]^{e_1} \cdots [a_r, b_r, c_r]^{e_r}, \quad e_j = \pm 1.$$

Hence, using Lemma 4.7 recursively, we obtain $\text{tr}'x^\sigma \equiv \text{tr}'x \pmod{J^2}$ for any $x \in F_n$. This completes the proof of Lemma 5.11. \square

Then we have

Theorem 5.12. *For any $n \geq 3$, $\mathcal{E}_{F_n}(1) = \text{Inn } F_n \cdot \mathcal{A}_{F_n}(2)$.*

Proof. Recall the argument in the former part of Lemma 5.9. For any $\sigma \in \mathcal{E}_{F_n}(1)$, there exists some $\iota \in \text{Inn } F_n$ such that

$$x^{\sigma\iota} \equiv x \pmod{\Gamma_{F_n}(3)}$$

for any $x \in F_n$. This shows that $\sigma\iota \in \mathcal{A}_{F_n}(2)$. This completes the proof of Theorem 5.12. \square

At the end of this subsection, we prove

Theorem 5.13. *For any $k \geq 1$, $\mathcal{A}_{F_n}(2k) \subset \mathcal{E}_{F_n}(k)$.*

Proof. For any $\sigma \in \mathcal{A}_{F_n}(2k)$ and $x \in F_n$, we have $x^\sigma = xc$ for some $c \in \Gamma_{F_n}(2k+1)$. By Lemma 3.1, the element c is written as

$$c = c_1^{e_1} c_2^{e_2} \cdots c_r^{e_r}$$

for some left-normed commutators c_i of weight $2k+1$ and $e_i = \pm 1$. Hence, from Proposition 4.9 and Corollary 4.10, we obtain

$$\text{tr}'x^\sigma \equiv \text{tr}'x \pmod{J^{k+1}}.$$

This shows that $\sigma \in \mathcal{E}_{F_n}(k)$. This completes the proof of Theorem 5.13. \square

5.3. Graded quotients $\text{gr}^k(\mathcal{E}_{F_n})$.

In this subsection, we study the graded quotients $\text{gr}^k(\mathcal{E}_{F_n}) := \mathcal{E}_{F_n}(k)/\mathcal{E}_{F_n}(k+1)$. Since each $\mathcal{E}_{F_n}(k)$ is a normal subgroup of $\text{Aut } F_n$, the group $\text{Aut } F_n$ naturally acts on $\text{gr}^k(\mathcal{E}_{F_n})$ by the conjugation from the right. Furthermore since $\{\mathcal{E}_{F_n}(k)\}$ is a central filtration, the action of $\mathcal{E}_{F_n}(1)$ on $\text{gr}^k(\mathcal{E}_{F_n})$ is trivial. Hence we can consider each $\text{gr}^k(\mathcal{E}_{F_n})$ as an $\text{Aut } F_n/\mathcal{E}_{F_n}(1)$ -module. Here, we introduce Johnson homomorphism like homomorphisms to study the $\text{Aut } F_n/\mathcal{E}_{F_n}(1)$ -module structure of $\text{gr}^k(\mathcal{E}_{F_n})$.

To begin with, for any $k \geq 1$ and $\sigma \in \mathcal{E}_{F_n}(k)$, define a map $\eta_k(\sigma) : \text{gr}^1(J) \rightarrow \text{gr}^{k+1}(J)$ by

$$\eta_k(\sigma)(f) := s_\sigma(f) = f^\sigma - f \in \text{gr}^{k+1}(J)$$

for any $f \in J$. The well-definedness of the map $\eta_k(\sigma)$ follows from Lemma 5.2. It is easily seen that $\eta_k(\sigma)$ is a homomorphism between abelian groups.

Then we have a map $\eta_k : \text{gr}^k(\mathcal{E}_{F_n}) \rightarrow \text{Hom}_{\mathbf{Z}}(\text{gr}^1(J), \text{gr}^{k+1}(J))$ defined by $\sigma \mapsto \eta_k(\sigma)$. For any $\sigma, \tau \in \mathcal{E}_{F_n}(k)$, from (1) of Lemma 5.1, and from Lemma 5.2, we see

$$s_{\sigma\tau}(f) = (s_\sigma(f))^\tau + s_\tau(f) \equiv s_\sigma(f) + s_\tau(f) \pmod{J^{k+2}}.$$

This shows that η_k is a homomorphism of abelian groups. By the definition, each of η_k is injective. Furthermore, we have

Lemma 5.14. *For each $k \geq 1$, η_k is an $\text{Aut } F_n / \mathcal{E}_{F_n}(1)$ -equivariant.*

Proof. It suffices to show that η_k is an $\text{Aut } F_n$ -equivariant. For any $\sigma \in \text{Aut } F_n$ and $\tau \in \mathcal{E}_{F_n}(k)$, we see

$$\begin{aligned} \eta_k(\tau \cdot \sigma)(f) &= \eta_k(\sigma^{-1}\tau\sigma)(f) = s_{\sigma^{-1}\tau\sigma}(f), \\ (\eta_k(\tau) \cdot \sigma)(f) &= (\eta_k(\tau)(f^{\sigma^{-1}}))^\sigma = s_\tau(f^{\sigma^{-1}})^\sigma = (f^{\sigma^{-1}\tau} - f^{\sigma^{-1}})^\sigma \\ &= f^{\sigma^{-1}\tau\sigma} - f = s_{\sigma^{-1}\tau\sigma}(f) \end{aligned}$$

for any $f \in J$. Hence we have $\eta_k(\tau \cdot \sigma) = \eta_k(\tau) \cdot \sigma$. This means η_k is an $\text{Aut } F_n$ -equivariant homomorphism. This completes the proof of Lemma 5.14. \square

Using the homomorphisms η_k , we see that $\text{gr}^k(\mathcal{E}_{F_n})$ is an $\text{Aut } F_n / \mathcal{E}_{F_n}(1)$ -submodule of the \mathbf{Q} -vector space $\text{Hom}_{\mathbf{Z}}(\text{gr}^1(J), \text{gr}^{k+1}(J))$, and hence we obtain

Theorem 5.15. *For any $n \geq 3$,*

- (1) *Each of $\text{gr}^k(\mathcal{E}_{F_n})$ is torsion-free.*
- (2) *$\dim_{\mathbf{Q}}(\text{gr}^k(\mathcal{E}_{F_n}) \otimes_{\mathbf{Z}} \mathbf{Q}) < \infty$.*

As a corollary to Theorem 5.13, we see that $\text{gr}^1(\mathcal{E}_{F_n})$ is finitely generated. In general, however, it seems to be quite a difficult to determine the $\text{Aut } F_n / \mathcal{E}_{F_n}(1)$ -module structure of $\text{gr}^k(\mathcal{E}_{F_n})$ even the case where $k = 1$.

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